Comment on Geometric Phases for Classical Field Theories

The geometric phase in an adiabatic cyclic evolution of a quantum system was discovered by Berry,[1] and generalized to nonadiabatic motions by Aharonov and Anandan.[2] Recently, Garrison and Chiao[3] have further extended it to classical nonlinear field theories, which are invariant under global gauge transformations. In this Comment I point out that the latter restriction can be removed and the geometric phase may be regarded as due to just a complex inner product on the Hilbert space $\mathcal{H}$, consisting of the set of L_2 integrable functions on $\mathbb{R}^3$, that need not be conserved. Also, I generalize it to a non-Abelian geometric phase which extends Anandan's[4] nonadiabatic generalization of the Wilczek-Zee[5] phase to classical fields that may be nonlinear. Suppose that the field $\psi(\mathbf{x},t)$ satisfies the wave equation

$$i\hbar\,\partial\psi(\mathbf{x},t)/\partial t = G(\psi(\mathbf{x},t),\nabla\psi(\mathbf{x},t),\psi^*(\mathbf{x},t),\nabla\psi^*(\mathbf{x},t)). \tag{1}$$

Consider a cyclic evolution in the interval $[0,\tau]$, by which I mean that $\psi(\mathbf{x},\tau)=e^{i\phi}\psi(\mathbf{x},0)$, where ϕ is a *complex* number. Write $\psi(\mathbf{x},t)=e^{if(t)}\tilde{\psi}(\mathbf{x},t)$, where $f(t)$ is a complex function and $\tilde{\psi}(\mathbf{x},\tau)=\tilde{\psi}(\mathbf{x},0)$. Then (1) yields

$$\phi=\int_0^\tau dt\,\frac{i(\tilde{\psi},\dot{\tilde{\psi}})}{(\tilde{\psi},\tilde{\psi})}-\int_0^\tau dt\,\frac{(\psi,G)}{\hbar(\psi,\psi)}, \tag{2}$$

where the parentheses represent the inner product and the dot denotes differentiation with respect to time. The last term in (2) is dynamical in the sense that it depends on G. But the preceding term, which I shall call β, is geometrical in the sense that it depends only on the projection $\hat{C}$ of the evolution on the projective Hilbert space[2] $\mathcal{P}$ of the rays of $\mathcal{H}$. $e^{i\beta}$ may be regarded as the holonomy transformation of a connection on the natural line bundle[2,6] over $\mathcal{P}$. Indeed, $e^{i\beta}$ is invariant if ψ is multiplied by a complex function $F(t)$ such that $F(\tau)=F(0)$. However, the imaginary part of β vanishes, so that

$$\beta=\oint_{\hat{C}}\frac{i(\tilde{\psi},d\tilde{\psi})-i(d\tilde{\psi},\tilde{\psi})}{2(\tilde{\psi},\tilde{\psi})}, \tag{3}$$

where d is the differential operator on $\mathcal{P}$. It is possible to choose $\tilde{\psi}$ to have unit norm, in which case β has the same form as in the previous work.[2] Hence the geometry is the same as in the previous treatments[1,2] in which the evolution of ψ was assumed to be linear and unitary. Thus the geometric phase is not only independent of the Hamiltonian that generates motion along a given closed curve in $\mathcal{P}$, but is also independent of whether the evolution is linear or unitary.

To observe this phase experimentally, it would be necessary to "turn off" nonlinearities so that ψ can be made to interfere with another field which has not undergone the same motion. An example, given by Garrison and Chiao,[3] is the propagation of a beam of elliptically polarized light in a Kerr-active medium. When the beam emerges from the medium it can be made to interfere with a coherent beam because linearity has been restored.

Suppose now that the Lagrangean density $\mathcal{L}$ has a U(n) gauge invariance. Let $\{\psi_a(\mathbf{x},t),\ a=1,2,\ldots,n\}$ be a set of orthonormal states related by the gauge group and which obey (1). Choose another basis $\{\tilde{\psi}_a(\mathbf{x},t)\}$ for the subspace $V_n(t)$ spanned by $\{\psi_a\}$ with $\tilde{\psi}(\mathbf{x},0)=\tilde{\psi}(\mathbf{x},\tau)$. Then $\psi_a=U_{ba}\tilde{\psi}_b$, where $U(t)$ is a unitary matrix and the summation convention is used. Also, denoting G in (1) with ψ_a ($\tilde{\psi}_a$) replacing ψ by G_a ($\tilde{G}_a$), we have

$$G_a=\partial\mathcal{L}/\partial\psi_a^*-\partial_i\,\partial\mathcal{L}/\partial(\partial_i\psi_a^*)=U_{ba}\tilde{G}_b \tag{4}$$

on use of the chain rule. From (1) and (4),

$$U(\tau)=P\exp\left\{\oint_{\hat{C}}dt\,i(A-K)\right\}, \tag{5}$$

where $A_{ab}=i(\tilde{\psi}_a,\dot{\tilde{\psi}}_b)$,

$K_{ab}=\hbar^{-1}(\tilde{\psi}_a,\tilde{G}_b)$,

and $\hat{C}$ is the closed curve determined by the evolution of $\{\psi_a(\mathbf{x},t)\}$ in the Grassmann manifold $\mathcal{G}$ consisting of the n-dimensional subspaces of $\mathcal{H}$. The non-Abelian connection on $\mathcal{G}$ that is represented by A extends the nonadiabatic generalization[4] of the Wilczek-Zee connection[5] to nonlinear fields.

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