Bifurcations to Local and Global Modes in Spatially Developing Flows

J. M. Chomaz, P. Huerre, and L. G. Redekopp
Department of Aerospace Engineering, University of Southern California, Los Angeles, California 90089
(Received 22 June 1987)

We outline a possible scenario of successive bifurcations to local and global modes on a Ginzburg-Landau model with varying coefficients. It is shown that self-sustained resonances may appear via a Hopf bifurcation when the system exhibits a region of local absolute instability which is sufficiently large. Our findings are in qualitative agreement with experimental observations of spatially developing flows such as wakes and inhomogeneous jets.

PACS numbers: 47.20.Ft, 47.20.Ky

A sequence of bifurcations to local and global modes is illustrated on the Ginzburg-Landau equation with varying coefficients. The results provide a qualitative explanation for the occurrence of hydrodynamic resonances in spatially developing shear flows such as wakes and inhomogeneous jets where the absolute or convective character of the instability mechanism is altered locally as a function of a control parameter.

Rigorous definitions of absolute and convective instability have been given in the context of plasma physics by Briggs[1] and Bers[2] and similar concepts have recently been applied to inviscid instabilities in shear flows.[3-14] It is assumed initially that the basic flow is *parallel*, i.e., independent of the streamwise coordinate x. Its linear instability properties can then be formally characterized by a dispersion relation of the form $D[k,\omega;\mu]=0$, where k, ω, and μ denote the wave number, frequency, and control parameter, respectively. The parallel flow is said to be *absolutely unstable* if the Green's function $G(x,t;\mu)$ associated with the operator $D[-i\partial/\partial x', i\partial/\partial t';\mu]$ is such that $G\rightarrow\infty$ for all x as $t\rightarrow\infty$. Conversely the parallel flow is said to be *convectively unstable* if $G\rightarrow 0$ for all x as $t\rightarrow\infty$. A relatively simple mathematical criterion[1,2] is available to determine the nature of the instability: The flow is convectively (absolutely) unstable when the singularities $\omega_0=\omega(k_0)$ such that $[\partial\omega/\partial k]_{k_0}=0$ lie in the lower (upper) half ω plane. In general, the points ω_0 are branch-point singularities of $k(\omega)$. As an additional requirement, the branches $k^+(\omega)$ and $k^-(\omega)$ pertaining to each branch point should originate from distinct upper and lower halves of the complex k plane as $\omega_i=\text{Im}\,\omega$ is decreased from positive values towards zero.

When the basic flow varies slowly along the streamwise direction, the dispersion relation becomes a function of a slow spatial scale $X=\mu' x$, with $\mu'\ll 1$, and the previous concepts then apply locally at *each station X*. For many nonparallel flows such as spatially developing mixing layers,[11,15] flat-plate boundary layers,[16] and homogeneous jets [see Fig. 1(a)], the mean flow is locally convectively unstable everywhere with respect to *vortical* fluctuations. In such systems, any initial disturbance is advected by the flow as it is amplified. The medium is extremely sensitive to external coherence forcing[17] and the flow can be thought of as a collection of spatially evolving vortical instability waves of different frequencies traveling in the downstream direction. Measured frequency spectra are generally broadband. Self-sustained oscillations do not seem to be possible in this class of flows although significant feedback effects[17] could be induced by *global* pressure fluctuations[18] present in the far-field region as sketched in Fig. 1(a).

In contrast, strong self-sustained oscillations can be obtained when, in the same developing free shear flow, one introduces a second streamlined or blunt body at a finite distance downstream [see Fig. 1(b)]. For particular values of the distance between the two objects, a resonance can be triggered which gives rise to monochromatic acoustic radiation known as wake tones, jet tones, or edge tones, depending on the specific shear-layer–solid-body configuration.[19] The flow is still locally convectively unstable from the point of view of vorticity fluctuations but the dynamics of the flow is dominated by a feedback loop [Fig. 1(b)]: The downstream branch con-

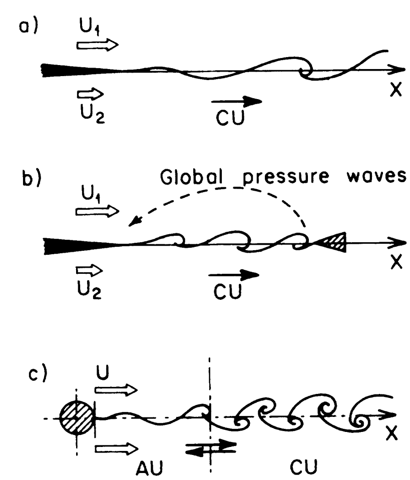

FIG. 1. Classes of spatially developing flows. (a) Extrinsic flows: no resonances. (b) Intrinsic flows: hydroacoustic resonances. (c) Intrinsic flows: hydrodynamic resonances.

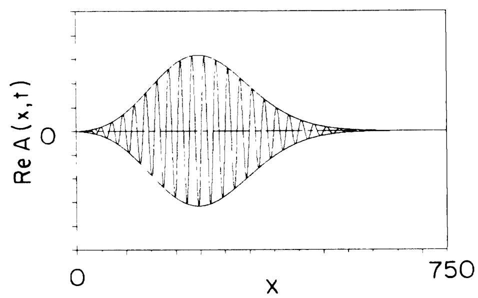

FIG. 2. Linear eigenfunction $A_1(x,t)$. Carrier wave: $\mathrm{Re}A_1(x,t)$. Envelope: $|A_1(x,t)|$. $\mu'=0.012$, $c_d=-10$. Shape independent of μ_0.

sists of *rotational instability waves* rolling up into vortices. The interaction between the vortical structures and the downstream body then generates *global* irrotational *pressure* disturbances which travel along the upstream branch of the loop. Resonance occurs when vortical shedding at the trailing edge of the upstream body is in phase with the vertical velocity induced by the global pressure fluctuations.

There is now increasing experimental and theoretical evidence that in wakes behind bluff bodies[3–9] and in inhomogeneous jets,[10,11] self-sustained oscillations can also be produced by purely hydrodynamic means, without the need for a second downstream body [Fig. 1(c)]. In this third class of spatially developing flows, the nature of the instability changes from locally absolute to locally convective at a particular downstream station X_t. This transition allows global oscillations of the separated flow to develop.[3] It is conjectured that the feedback loop is made up of temporally growing *vorticity waves* propagating in both flow directions.[3] The observed flow is then the finite-amplitude saturated state associated with these waves. It is found experimentally that such flows are relatively insensitive to infinitesimal external perturbations: Discrete frequency spectra are obtained with one or several fundamental components and their harmonics. The archetype of such a flow regime is the von Kármán vortex street behind a circular cylinder at low Reynolds numbers.

The Ginzburg-Landau equation arises in many weakly nonlinear studies of fluid dynamical systems close to marginal stability.[20,21] It serves as a simple model of the nonlinear evolution of hydrodynamic instability waves.[22,23] Furthermore, it is particularly well suited for our purpose: The associated dispersion relation is endowed with the minimum structure required for an algebraic branch point of order 2. In this context, the complex amplitude function $A(x,t)$ characterizing the spatiotemporal modulations of the marginal wave satisfies

$$\frac{\partial A}{\partial t}+U\frac{\partial A}{\partial x}=\mu A+(1+ic_d)\frac{\partial^2 A}{\partial x^2}-(1+ic_n)|A|^2A, \quad (1)$$

where U, μ, c_d, and c_n are given real coefficients.[24] Here the cubic nonlinearity is chosen to be stabilizing as in the case of a supercritical Hopf bifurcation and in consistency with experimental results for wakes behind circular cylinders.[5,6] The operator (1) linearized around the equilibrium solution $A=0$ admits a single temporal mode $\omega(k)=\omega_0+(c_d-i)(k-k_0)^2$, and the corresponding spatial solution $k(\omega)$ exhibits a branch point at the particular value $k_0=-U/[2(c_d-i)]$, $\omega_0=i\{\mu-U^2[4(1+ic_d)]\}$, where the group velocity $[d\omega/dk]_{k_0}$ becomes zero. For fixed settings of U and c_d, the parameter μ determines the nature of the instability as dictated by the sign of $\mathrm{Im}\omega_0$. The values $\mu=0$ and $\mu=\mu_t=U^2[4(1+c_d^2)]$ are of special interest. When $\mu<0$, the solution $A=0$ is linearly stable. In the range $0<\mu<\mu_t$ it becomes convectively unstable. Finally, the instability is absolute as soon as $\mathrm{Im}\omega_0$ becomes positive, i.e., when $\mu>\mu_t$.

Resonances are now sought in situations where U, c_d, and c_n remain constant, but where the control parameter μ is allowed to vary linearly on the interval $0<x<\infty$, according to $\mu=\mu_0+\mu' x$ with $\mu'<0$. This simple problem mimics in a very idealized manner the spatially developing flows depicted in Figs. 1(a) or 1(c). The parameter μ plays the role of a local Reynolds number based on, say, the local thickness of the wake while μ_0 is analogous to a global bifurcation parameter such as the Reynolds number based on cylinder diameter. The presence of the body is very crudely modeled by the boundary condition $A(x=0)=0$. Furthermore, since $\mu'<0$, the solution $A=0$ is linearly stable for sufficiently large x and we may impose the boundary condition $A(\infty)=0$. This latter feature is consistent with observations of wakes behind circular cylinders at low Reynolds numbers[25] (below 200) which indicate that the von Kármán vortex street decays far downstream.

The linearized Ginzburg-Landau equation for varying μ, together with the indicated boundary conditions, admits a denumerable set of solutions of the form $\phi_n(x)e^{-i\omega t}$ with

$$\omega_n=i[\mu_0-U^2/4(1+ic_d)+\{(1+ic_d)\mu'^2\}^{1/3}\zeta_n], \quad (2)$$

$$\phi_n(x)=\mathrm{Ai}([-\mu'/(1+ic_d)]^{1/3}x+\zeta_n), \quad (3)$$

where Ai denotes the usual Airy function and ζ_n its countable set of zeros. The largest temporal growth rate $\mathrm{Im}\omega_n$ is obtained for the mode $n=1$ which therefore dominates the linear regime. A typical eigenfunction is represented in Fig. 2. There exists a critical value

$$\mu_c=\mu_t+(-\zeta_1)|\mu'|^{2/3}(1+c_d^2)^{1/6}\cos[\tfrac{1}{3}\tan^{-1}c_d], \quad (4)$$

below which all the eigenvalues ω_n are such that $\mathrm{Im}\omega_n<0$ for all n. In the range $\mu_0<\mu_c$ all possible global modes are therefore necessarily damped. When $\mu_0>\mu_c$ the temporal growth rate $\mathrm{Im}\omega_1$ becomes positive and a supercritical Hopf bifurcation takes place whereby a complex pair of eigenvalues crosses the real-ω axis. As μ_0 increases above μ_c the number of modes giving rise to amplified resonances also increases.

Until now, only *global* instability concepts on the line $x>0$ have been invoked to examine the occurrence of self-sustained oscillations. For small values of μ', however, one may legitimately apply the WKB approximation and appeal to *local* instability arguments. In the sequel μ' is deliberately chosen to be small so as to allow both local and global descriptions to be relevant. The coefficient μ_0 is used as a control parameter, all other parameters remaining fixed. In all cases considered, $1+c_d c_n>0$ so that uniform Stokes wave-train solutions are linearly stable with respect to the Benjamin-Feir instability mechanism.[26] To elucidate the spatiotemporal development of disturbances, the nonlinear Ginzburg-Landau equation was solved numerically by finite differences, with white-noise conditions.

A simple scenario emerges as μ_0 is varied (see Fig. 3). When $\mu_0<0$ [Fig. 3(a)], $\mu(x)$ is always negative and $A=0$ is locally linearly stable everywhere. Global solutions are also damped and one concludes that $A=0$ is both locally and globally stable. As μ_0 crosses zero [Fig. 3(b)], a region of local convective instability develops close to the origin. This results in the appearance of a bulge of local growth in the amplitude evolution of the waves. Ultimately, however, disturbances decay and $A=0$ is globally stable. In the range $0<\mu_0<\mu_t$ no self-sustained oscillations are therefore possible. Nonetheless, the system is locally convectively unstable and the application of a continuous excitation within the convectively unstable region will lead to spatial growth of fluctuations. Thus the response of the system is seen to be *extrinsic*: It depends on the nature of external forcing, in qualitative agreement with what is known of convectively unstable open flows such as boundary layers, shear layers, etc. The parameter value $\mu_0=0$ may be referred to as a point of *local bifurcation.*

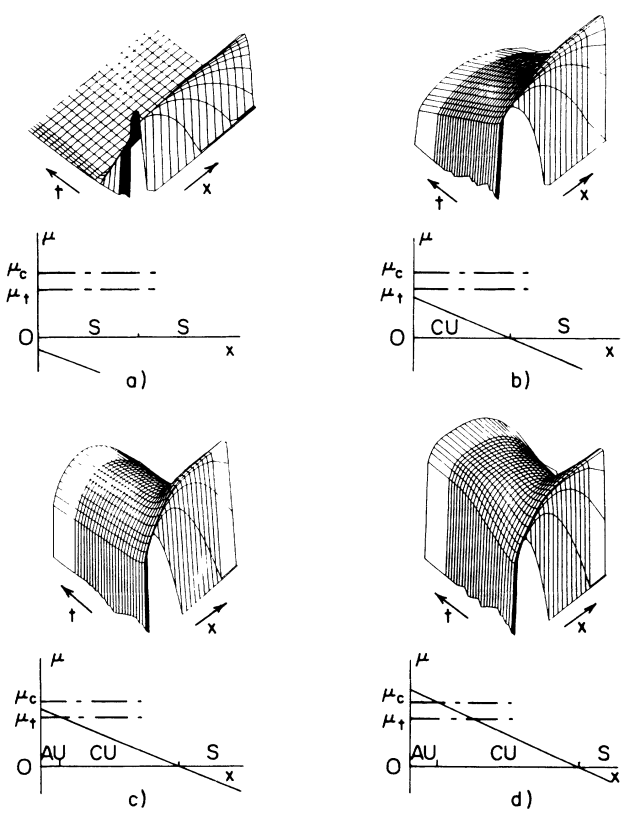

FIG. 3. Spatiotemporal evolution of impulse resonance $\mathrm{Im}|A(x,t)|$, as μ_0 increases. $\mu'=0.012$, $c_d=-10$, $c_n=0$, $U=6$. (a) $\mu_0<0$. (b) $0<\mu_0<\mu_t$. (c) $\mu_t<\mu_0<\mu_c$. (d) $\mu_0>\mu_c$.

As μ_0 increases above μ_t, a pocket of absolute instability develops near the origin. From (4), one notes that $\mu_c>\mu_t$. In other words, the value $\mu_0=\mu_t$ at which a region of local absolute instability appears is lower than the bifurcation value $\mu_0=\mu_c$ which signals the onset of global oscillations. Thus there exists a range $\mu_t<\mu_0<\mu_c$ where the solution $A=0$ becomes locally absolutely unstable but where global resonances cannot be sustained. It is worth noticing that the spatial extent of the domain of absolute instability can be as large as

$$x_T=-\zeta_1(-\mu')^{1/3}(1+c_d^2)^{1/6}\cos[\tfrac{1}{3}\tan^{-1}c_d]$$
$$=O(|\mu'|^{-1/3}), \qquad (5)$$

without there being self-sustained resonances. This regime is therefore characterized by transient local growth fluctuations [Fig. 3(c)], as in the previous range $0<\mu_0<\mu_t$.

Finally, when μ_0 exceeds μ_c [Fig. 3(d)], a Hopf *bifurcation* to a global mode takes place whereby the solution $A=0$ becomes globally unstable: The eigenvalue ω_1 acquires a positive imaginary part and the nonlinear regime is described near μ_c by the classical Landau normal form. The system exhibits an *intrinsic* mode of oscillation and the effect of small external forcing at an arbitrary frequency is likely to be small. For a near-resonant excitation, the Hopf bifurcation to a global mode is perfect.[27] This regime serves as a model of open flows with a sufficiently large region of absolute instability such as the von Kármán vortex street behind bluff bodies.[3-9]

We note that the region of absolute instability allows, for *positive* values of $\mathrm{Im}\omega$, a smooth switching[14] between the spatial branches $k^+(\omega)$ and $k^-(\omega)$ as x increases. This is, however, not sufficient to enforce the boundary conditions, which are only satisfied for discrete values of ω. Thus the existence of a pocket of absolute instability is a *necessary* but not sufficient condition for the onset of amplified global oscillations. The region of absolute instability must reach a critical size in order for resonances to occur. These findings suggest that resonant behavior in spatially developing flows cannot be attributed solely to the appearance of a region of absolute instability.

The qualitative scenario outlined above is consistent with recent experiments on wakes behind circular cylinders[5-7] and with linear stability analyses of the local velocity profiles.[3,4,7-9] When the Reynolds number $N_{\rm Re}$ based on cylinder diameter is below a value $N_{{\rm Re},n}$ of the order of 20 (corresponding in our model to $\mu_0=0$), the wake is both locally and globally stable. Parallel stability analyses[8] indicate that the local velocity profiles first become convectively unstable at extremely low local Reynolds numbers of the order of 7 (based on wake width). Correspondingly, as $N_{\rm Re}$ exceeds $N_{{\rm Re},n}$ a local bifurcation takes place whereby a region of convective instability appears behind the cylinder.[6,7] No von Kármán vortex street can be detected, however, unless coherent external forcing is applied close to the frequency of the damped resonance.[5,28] At a critical value $N_{{\rm Re},c}$ of the order of 47, a Hopf bifurcation to a global mode occurs. The validity of the Landau normal form and the scaling laws which it implies when $|(N_{\rm Re}-N_{{\rm Re},c})/N_{{\rm Re},c}|\ll 1$ have been verified experimentally.[5,6] Furthermore, when $N_{\rm Re}>N_{{\rm Re},c}$, the region of absolute instability coincides approximately with the recirculation eddies behind the cylinder.[6] At very high Reynolds numbers, the pocket of absolute instability subsists[7] but the wake is then fully turbulent.

Finally, this investigation suggests that some caution needs to be exercised when one is seeking frequency-selection criteria for global resonance in spatially developing flows. Various proposals have been put forward which are all based on *local* stability arguments. For instance, one may choose as preferred frequency $\omega_{0,r\max}$, i.e., the real part associated with the maximum absolute growth rate $\omega_{0,i\max}$ over the entire domain.[14] An alternative selection mechanism proposed by Koch[3] leads to the value of ω_0 at $\mu=\mu_t$, i.e., at the location separating the absolutely unstable region from the convectively unstable one. In the present model, restricted as it is to varying $\mu(x)$ only, both local criteria give the same result: $\omega_{\rm local}=U^2c_d/[4(1+c_d^2)]$. The exact expression (2) from global linear theory yields

$$\mathrm{Re}\,\omega_1=\omega_{\rm local}-\zeta_1(1+c_d^2)^{1/6}(-\mu')^{2/3}\cos[\tfrac{1}{3}\tan^{-1}c_d].$$

Thus local criteria fail to account for the second term which, in the WKB approximation, is $O((-\mu')^{2/3})$. It therefore appears that the local criterion of Pierrehumbert[14] provides a leading-order estimate of the global frequency. Higher-order terms in the WKB parameter include the effect of boundary conditions.

The authors wish to thank Pierre Monkewitz for many very helpful and stimulating discussions. This work is supported by the U.S. Air Force Office of Scientific Research under Grant No. F49620-85-C-0080.

[1]R. J. Briggs, *Electron-Stream Interaction with Plasmas* (MIT Press, Cambridge, MA, 1964).

[2]A. Bers, in *Physique des Plasmas,* edited by C. DeWitt and J. Peyraud (Gordon and Breach, New York, 1975).

[3]W. Koch, J. Sound Vib. **99**, 53 (1985).

[4]P. A. Monkewitz and L. N. Nguyen, J. Fluids Struct. **1**, 165 (1987).

[5]M. Provansal, C. Mathis, and L. Boyer, "Bénard–von Kármán Instability: Transient and Forced Regimes" (to be published).

[6]P. J. Strykowski, Ph.D. thesis, Yale University, 1986 (unpublished).

[7]G. M. Triantafyllou, M. S. Triantafyllou, and C. Chryssostomidis, J. Fluid Mech. **170**, 461 (1986).

[8]P. A. Monkewitz, "Absolute and Convective Nature of the Instability in Two-Dimensional Wakes at Low Reynolds Numbers" (to be published).

[9]L. S. Hultgren and A. K. Aggarwal, "A Note on Absolute Instability of the Gaussian Wake" (to be published).

[10]P. A. Monkewitz and K. D. Sohn, American Institute of Aeronautics and Astronautics Report No. 86-1882, 1986 (unpublished).

[11]P. Huerre and P. A. Monkewitz, J. Fluid Mech. **159**, 151 (1985).

[12]P. Huerre, in *Instabilities and Nonequilibrium Structures,* edited by E. Tirapegui and D. Villaroel (Reidel, Dordrecht, The Netherlands, 1987), p. 141.

[13]L. Merkine, Geophys. Astrophys. Fluid Dyn. **9**, 129 (1977).

[14]R. T. Pierrehumbert, J. Atmos. Sci. **41**, 2142 (1984).

[15]T. F. Balsa, J. Fluid Mech. **174**, 553 (1987).

[16]M. Gaster, Proc. Roy. Soc. London A **347**, 271 (1975).

[17]C. M. Ho and P. Huerre, Annu. Rev. Fluid Mech. **16**, 365 (1984).

[18]D. G. Crighton and P. Huerre, American Institute of Aeronautics and Astronautics Report No. 84-2295, 1984 (unpublished).

[19]D. Rockwell and E. Naudascher, Annu. Rev. Fluid Mech. **11**, 67 (1979).

[20]A. C. Newell and J. A. Whitehead, J. Fluid Mech. **38**, 279 (1969).

[21]K. Stewartson and J. T. Stuart, J. Fluid Mech. **48**, 529 (1971).

[22]R. J. Deissler, J. Stat. Phys. **40**, 371 (1985).

[23]R. J. Deissler, Los Alamos National Laboratory Report No. LA-UR-85-4211, 1985 (unpublished).

[24]The term involving U cannot be removed by a Galilean transformation when boundary conditions at fixed spatial positions are imposed.

[25]D. Desruelle, M.S. thesis, Illinois Institute of Technology, 1983 (unpublished).

[26]J. T. Stuart and R. C. DiPrima, Proc. Roy. Soc. London A **362**, 27 (1978).

[27]J. Guckenheimer and P. J. Holmes, *Nonlinear-Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer-Verlag, Berlin, 1983).

[28]M. Nishioka and H. Sato, J. Fluid Mech. **89**, 49 (1978).