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Quantum Correlations: A Generalized Heisenberg Uncertainty Relation

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A theoretical result for correlated quantum systems is presented which leads to a noise commutation relation and a generalized Heisenberg uncertainty relation. These relations imply an inherent and unavoidable extra noise in quantum measurements beyond that already included in the Heisenberg lower bound. These relations lead directly to model-independent lower bounds on inherent noise, useful in a variety of applications, including balanced homodyne detection and quantum optical linear amplifiers.

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Recent advances, particularly in the area of quantum optics, have caused heightened interest in the fundamental limitations on the achievable accuracy that may be obtained in the measurement of quantum-mechanical systems. In this Letter we present a quantum correlation result and a noise commutation relation for correlated quantum systems. The measurement of quantum-system observables requires the correlation of these "micro-observables" and the measuring apparatus (usually a macroscopic system). The noise commutation relation is applied to quantum measurement leading to a generalized Heisenberg uncertainty relation. The generalized Heisenberg uncertainty relation yields a lower bound on the inherent, irreducible, *extra noise* in quantum measurements, which is due to the measuring process itself.[1] For example, Arthurs and Kelly[2] introduced theoretically a complete joint measurement of noncommuting observables which showed the state preparation of "unbalanced," or "squeezed," states and exhibited the extra noise implied by the generalized Heisenberg uncertainty relation.

To indicate the utility of the noise commutation relation, two different applications are discussed. First, an application of the generalized Heisenberg uncertainty relation indicates how a model-independent noise limit for balanced homodyne detection can be obtained. A different application is discussed in which the noise commutation relation is applied to develop a model-independent lower bound for the inherent noise of a quantum optical linear amplifier.

We start with two quantum systems, 1 and 2. The overall quantum system is represented on the tensor product Hilbert space $\mathbf{H}=H_1\otimes H_2$. The observables on the individual spaces are represented by the operators $\hat{A}_1,\hat{B}_1,\ldots,$ and $\ldots,\hat{Y}_2,\hat{Z}_2$ where operators are indicated by a caret, and the subscripts indicate the Hilbert space associated with that operator (we use the Heisenberg picture throughout this paper). These definitions may be extended to observables on the tensor product space written as $\hat{A}=\hat{A}_1\otimes\hat{I}_2$ and $\hat{Y}=\hat{I}_1\otimes\hat{Y}_2$ and so on, where $\hat{I}_1$ and $\hat{I}_2$ represent the identity operators on H_1 and H_2. The state of the system is given by the statistical operator $\hat{\rho}$, a nonnegative Hermitean operator, defined by

$$\hat{\rho}=\hat{\rho}_1\otimes\hat{\rho}_2,$$

with $\mathrm{Tr}\hat{\rho}=1$ (i.e., the two systems are independent).

Now consider a system observable $\hat{A}=\hat{A}_1\otimes\hat{I}_2$. We allow a quantum correlation of this system observable with another observable $\hat{Y}$. Further, we define a "noise operator" $\hat{N}_Y$, which indicates how closely $\hat{Y}$ "tracks" $\hat{A}$:

$$\hat{N}_Y=\hat{Y}-G_Y\hat{A}, \tag{1}$$

where G_Y is a real constant. We require that the quantum correlation have a form such that the "tracking" observables match, on average, the system observables. That is,

$$\mathrm{Tr}(\hat{\rho}\hat{N}_Y)=0. \tag{2}$$

Now, if (2) is true uniformly for all states $\hat{\rho}_1$ (for fixed $\hat{\rho}_2$), then it can be shown[1] that

$$\mathrm{Tr}(\hat{\rho}\hat{N}_Y\hat{C})=0 \tag{3}$$

for all Hermitian operators $\hat{C}=\hat{C}_1\otimes\hat{I}_2$ on **H**. That is, for tracking quantum correlations [in the sense of Eq. (2)], the noise operator $\hat{N}_Y$ is *uncorrelated* with *all* system operators $\hat{C}$ on **H**. We will show that this simple result has interesting physical consequences.

If we assume an initial system state $\hat{\rho}$ and a pair of system observables $\hat{A}$ and $\hat{B}$, which are tracked by observables $\hat{Y}$ and $\hat{Z}$, i.e.,

$$\mathrm{Tr}(\hat{\rho}\hat{N}_Y)=\mathrm{Tr}(\hat{\rho}\hat{N}_Z)=0$$

uniformly for all state vectors $\hat{\rho}_1$, then using (3), it can be shown that the rms deviations of $\hat{Y}$ and $\hat{Z}$, σ_Y and σ_Z, are given by

$$\sigma_Y^2=G_Y^2\sigma_A^2+\sigma_{N_Y}^2 \tag{4}$$

and

$$\sigma_Z^2=G_Z^2\sigma_B^2+\sigma_{N_Z}^2. \tag{5}$$

Calculating the expectation of the commutator $[\hat{N}_Y,\hat{N}_Z]$, we are led to the following result:

$$\mathrm{Tr}(\hat{\rho}[\hat{N}_Y,\hat{N}_Z]) =\mathrm{Tr}(\hat{\rho}[\hat{Y},\hat{Z}])-G_YG_Z\,\mathrm{Tr}(\hat{\rho}[\hat{A},\hat{B}]). \tag{6}$$

In the following sections we demonstrate some consequences of these results (4)-(6).

Following the treatment of von Neumann[3] and, more recently, Wigner,[4] we discuss the quantum measurement from the viewpoint of measuring pairs of observables. Using the formalism described above, we associate the quantum-system observables that are to be measured at a specific time (e.g., $t=0$) with system 1, and the measuring apparatus (or meters) with the system 2. In order to effect a measurement, the system observables $\hat{C}=\hat{C}_1\otimes\hat{I}_2$ and $\hat{D}=\hat{D}_1\otimes\hat{I}_2$ must be coupled to the measuring apparatus represented by the observables $\hat{R}=\hat{I}_1\otimes\hat{R}_2$ and $\hat{S}=\hat{I}_1\otimes\hat{S}_2$ for an interval of time t. (Here the $\hat{C}$'s are identified with the $\hat{A}$'s above, and the $\hat{R}$'s are identified with the $\hat{Y}$'s above.) A necessary condition for a meter $\hat{R}$ on H_2 to make a measurement of a system observable $\hat{C}$ on H_1 is that the meter tracks the system observable uniformly, for all initial states $\hat{\rho}_1$ of the system; that is, that [following Eqs. (1) and (2) above] $\langle\hat{N}_R\rangle=\langle\hat{N}_S\rangle=0$, where

$$\hat{N}_R=\hat{R}(t)-G_R\hat{C}(0),\quad \hat{N}_S=\hat{S}(t)-G_S\hat{D}(0),$$

and where the G's may be identified as the amplification gain between the system observables and the measuring apparatus. At some point in the measurement, information is passed to a macroscopic or "classical meter." We assume now that $\hat{R}$ and $\hat{S}$ are macroscopic measuring apparatus (i.e., $[\hat{R},\hat{S}]=0$), measuring $\hat{C}$ and $\hat{D}$. Starting with the noise commutator $[\hat{N}_R,\hat{N}_S]$ we have from Eq. (6)

$$|\mathrm{Tr}(\hat{\rho}[\hat{N}_X,\hat{N}_Y])|=|G_XG_Y\,\mathrm{Tr}(\hat{\rho}[\hat{A},\hat{B}])|. \tag{7}$$

Squaring this and using results (4) and (5) yields

$$\sigma_R^2\sigma_S^2=\sigma_C^2\sigma_D^2+\sigma_{N_R}^2\sigma_{N_S}^2+\sigma_C^2\sigma_{N_S}^2+\sigma_D^2\sigma_{N_R}^2. \tag{8}$$

We define the normalized meter operators by $\chi=\hat{R}/G_R$ and $\eta=\hat{S}/G_S$. From the Heisenberg uncertainty relation, we have that $\sigma_C^2\sigma_D^2\geq\frac{1}{4}|\mathrm{tr}(\hat{\rho}[\hat{C},\hat{D}])|^2$ and, similarly, $\sigma_{N_R}^2\sigma_{N_S}^2\geq\frac{1}{4}|\mathrm{tr}(\hat{\rho}[\hat{N}_R,\hat{N}_S])|^2$. Using these bounds and substituting into (8), we have

$$\sigma_\chi^2\sigma_\eta^2\geq|\mathrm{Tr}(\hat{\rho}[\hat{C},\hat{D}])|^2, \tag{9}$$

which is a generalized Heisenberg uncertainty relation in that the right-hand side of Eq. (9) is 4 times the Heisenberg uncertainty lower bound for $\hat{C}$ and $\hat{D}$ which addresses the uncertainty in the system observables, and not in the *measurement*. This additional noise is fundamental and unavoidable, since it depends on the measurement process itself and not on the apparatus details; further, it has important physical consequences, which will be indicated below.

To demonstrate the power of these results, we briefly address two applications: first, a limit on balanced homodyne detection; second, an application to determine a model-independent noise limit of a quantum optical linear amplifier.

An example of the type of paired observable measurement indicated in Eq. (9) above occurs in balanced homodyne detection of an optical signal which is used in the experimental detection of squeezed states of light, and coherent communications. For balanced homodyne detection, the observables $\hat{C}$ and $\hat{D}$ can be associated with the electromagnetic field quadratures, with

$$\hat{C}=\tfrac{1}{2}(\hat{a}+\hat{a}^\dagger);\quad \hat{D}=-\tfrac{1}{2}i(\hat{a}-\hat{a}^\dagger),$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators of the field mode and with $[\hat{C},\hat{D}]=\frac{1}{2}i$. By application of the generalized Heisenberg uncertainty relation (9), the lower bound on the uncertainty product for the measurement, $\sigma_\chi^2\sigma_\eta^2\geq\frac{1}{4}$ can immediately be determined. A model-specific analysis with 100% quantum efficiency[5] confirms the generalized Heisenberg uncertainty result.

Another example, an application of the noise commutation relation, is the calculation of the lower bound on the inherent noise produced in a quantum optical linear amplifier. In the theory of the quantum optical linear amplifier, gain is produced through the coupling of an optical mode to, for example, an inverted population of atoms.[6] We let $\hat{a}$ and $\hat{a}^\dagger$ represent the annihilation and creation operators for a mode with the Hamiltonian $\hat{H}=\hbar\omega(\hat{a}^\dagger\hat{a}+\frac{1}{2})$. Since these are not Hermitian operators, we form the pair $\hat{P}=(\hbar\omega/2)^{1/2}(\hat{a}+\hat{a}^\dagger)$ and $\hat{Q}=-i(\hbar/2\omega)^{1/2}(\hat{a}-\hat{a}^\dagger)$ so that $[\hat{Q},\hat{P}]=i\hbar$ and the energy is just $\langle \hat{H} \rangle = \frac{1}{2}(\langle \hat{P}^2 \rangle + \omega^2 \langle \hat{Q}^2 \rangle$. In this case, $\hat{Q}(0)$ and $\hat{P}(0)$ can be associated with $\hat{A}$ and $\hat{B}$ above, and the amplified signals with gain G, $\hat{Q}(t)$ and $\hat{P}(t)$, can be associated with $\hat{Y}$ and $\hat{Z}$ above. In the optical linear amplifier the output signals track the input signals in the sense of (1) and (2) above. Then $\hat{Q}(t) = \sqrt{G}\hat{Q}(0) + \hat{N}_Q$ and $\hat{P}(t) = \sqrt{G}\hat{P}(0) + \hat{N}_P$ and with the fact that the $[\hat{Q},\hat{P}]$ commutator is time invariant the Hamiltonian may be written as

$$\langle \hat{H} \rangle = G\langle \hat{P}(0) \rangle + G\langle \hat{Q}(0) \rangle + \langle \hat{E}_X \rangle,$$

where $\hat{E}_x$ is the extra noise present, above the gain amplified input noise. Then from the noise commutator (5),

$$\sigma_{N_Q}^2 \sigma_{N_P}^2 \geq \tfrac{1}{4}\hbar^2 |G-1|^2. \tag{10}$$

The extra noise is, however, just $\hat{E}_x = \frac{1}{2}(\sigma_{N_Q}^2 \omega^2 + \sigma_{N_P}^2)$. This is minimized when the two terms on the right-hand side are equal, and thus from (10) we have that the model-independent, extra noise in the quantum optical linear amplifier is just

$$\langle \hat{E}_x \rangle \geq \tfrac{1}{2}\hbar\omega |G-1|. \tag{11}$$

This lower bound is consistent with model-specific analyses: for example, Yamamoto and Haus[7] and Glauber.[6]

Heisenberg, through his famous microscope *Gedankenexperiment*,[8] showed that experiments may *inherently* affect the precision with which observables can be determined, a radical departure from classical physics. However, the uncertainty principle only addresses the necessary dispersion in the system observables prior to the measurement. We have shown that a lower bound that is 4 times the Heisenberg bound on the variances is the best that can be obtained when the *measuring apparatus* is considered as well. The noise commutation result and the generalized uncertainty relation can be used in a variety of applications to obtain fundamental lower limits in a model-independent manner.

[1]A more complete discussion of these results, including derivations, will be presented elsewhere: E. Arthurs and M. S. Goodman, to be published.

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