Critical Exponents and Scaling Relations for Self-Organized Critical Phenomena

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(Received 4 March 1988)

Critical indices β, γ δ, ν, etc. are defined and calculated for self-organized critical phenomena. Scaling relations are derived and checked numerically. The order-parameter exponent β describes the spontaneous current and the relaxation to the critical point. The power spectrum has "$1/f$" behavior with exponent $\phi=\gamma/\nu z$, where z is the dynamical critical exponent.

PACS numbers: 05.40.+j, 05.70.Jk, 05.70.Ln

Many extended dissipative dynamical systems evolve into structures with long-range fractal[1] spatial correlations, or long-range temporal correlations with "$1/f$" power spectrum.[2] It was suggested that this behavior may be caused by the self-organization of the systems into a "critical state."[3] The dynamical process creates a stationary state, where transport takes place through events on all length scales and all time scales: stationarity implies criticality. As examples of physical systems one might think of sand running through the hourglass, raindrops running down a window pane, light from quasars, motion of dislocations in a resistor, or even interactive economical systems. The phenomenon was demonstrated on a number of simple "cellular automata" models which were allowed to grow (or relax) to their stationary state. Perturbations, localized in time and space, yield responses on all length and time scales, with $1/f$ power spectrum $S(\omega)\approx\omega^{-\phi}$.

In this Letter we clarify the analogy with traditional critical phenomena[4] by defining several critical exponents [Eqs. 2(a)–2(f)]. We stress that although the notation is the same, the physics is entirely different: In statistical mechanics the exponents describe equilibrium static properties; here, they describe nonequilibrium dynamical properties. In addition to exponents characterizing the properties at criticality, we also introduce exponents ν, β, γ, etc., for the system when forced away from the critical point. The "order-parameter" exponent β describes both the spontaneous current above the critical point and the relaxation towards the critical point. Below the critical point, the relaxation is a stretched exponential, as found experimentally in glasses. The exponents are related through scaling relations [Eqs. (3), (5), (6)], also in analogy with the static critical phenomena.[4] Of particular interest is a relation [Eq. (5)] between the exponent ϕ of the power spectrum and the other critical exponents. Stationarity implies a scaling relation [Eq. (6)] which simply reads $\gamma/\nu=2$. This leads to an extremely simple result for the noise exponent: $\phi=2/z$, where z is the dynamical critical exponent. Numerical estimates on models in two and three dimensions are consistent with the scaling relations. A mean-field theory gives $\beta=1$, $\delta=2$, $\gamma=1$, $\nu=\frac{1}{2}$, $z=2$, and $\phi=1$. A more detailed discussion along with the description of numerical measurements of the exponents will be published elsewhere.[5]

The fundamental physics of the self-organized critical state is quite simple. To visualize, think of building a sandpile by adding particles randomly and very slowly. At the beginning, the pile is quite flat. The addition of a single particle causes (at most) a small local rearrangement. As we continue, the pile gets steeper and steeper, and the "avalanches" become larger and larger. Finally the pile will reach a statistically stationary state, where the rearrangements take place on any length scales and time scales, limited only by the size of the system. The particle flows caused by a single avalanche (a "cluster") obey a power-law distribution[3,6] $D(s)\approx s^{-\tau+1}$. The duration t of the avalanches obeys a similar distribution $D(t)\approx t^{-b}$, and $\phi=2-b$. This is the self-organized critical state.

The critical state can be reached either by our adding particles slowly, and allowing particles to leave the system at the boundary ("model 1") or by our gradually tilting an originally flat sandbox ("model 2"). We can also approach the critical state from the other side by starting with a large slope and letting the pile relax. So the self-organized critical state is an "attractor" for the dynamics. We emphasize that the sand picture is only a vivid example; the concept of self-organized criticality is obviously much more general.

Specific cellular-automaton models are defined in Ref. 3. The dynamics is very simple: If the local slope or pressure $z_{i,j}$ exceeds the critical value z_c, then at the next time step (in two dimensions)

$$z_{i,j}\to z_{i,j}-4,\quad z_{i,j\pm1}\to z_{i,j\pm1}+1,$$

$$z_{i\pm1,j}\to z_{i\pm1,j}+1.$$

In model 2 (to be studied here) the slope is increased by repeatedly letting $z_{i,j}\to z_{i,j}+1$ at random sites (i,j) and allowing the system to relax following the dynamical rule above. The boundary condition is $z=0$.

In order to define critical exponents, let us first identify the order parameter. If the "slope" $\theta = \langle z \rangle$ of the system is, somehow, kept larger than the "critical slope" θ_c, there will be a continuous "spontaneous flow" j. On the other hand, if $\theta \leq \theta_c$ there will be a flow j only when an "external field" is applied by our addition of particles or an increase in the pressure.

Thus, the analogy with "traditional" critical phenomena is now clear. The flow j is the "magnetization" or order parameter. The "magnetic field" h is the current of incoming particles (for model 1) or the rate of slope increase (for model 2). The deviation $\theta_c - \theta$ from the critical slope plays the role of the reduced temperature (or the deviation from the critical concentration for a percolation transition). A lower slope ($\theta < \theta_c$) can be achieved by our stopping the buildup of the system before it reaches criticality, or by our lowering the slope once the system reaches criticality. A high slope ($\theta > \theta_c$) can be achieved by the application of a finite field h to the system,[7] and a wait for stationarity.

The susceptibility χ characterizes the response to the field:

$$\delta j(x,t) = \int\int \chi(x,x';t,t')\delta h(x',t')dx'dt'. \tag{1}$$

The correlation length ξ is the cutoff in linear cluster size below criticality, which is related to the cutoff in cluster size through the fractal dimension D, $s_{co} \approx \xi^D$. Thus, we conjecture the following power laws for the average quantities:

$$j \approx (\theta - \theta_c)^\beta, \tag{2a}$$

$$\chi \approx (\theta_c - \theta)^{-\gamma}, \tag{2b}$$

$$\xi \approx (\theta_c - \theta)^{-\nu}, \tag{2c}$$

$$s_{co} \approx (\theta_c - \theta)^{-1/\sigma}, \tag{2d}$$

and

$$j(\theta = \theta_c) \approx h^{1/\delta}. \tag{2e}$$

A dynamical exponent z relates the relaxation time t to the linear size l of the cluster:

$$t \approx l^z, \quad 1 \lesssim l \lesssim \xi. \tag{2f}$$

We now derive scaling relations by means of the cluster picture rather than postulating scaling functions, in order to connect with the traditional picture of $1/f$ noise as a superposition of events with power-law temporal distribution. First, the flow j caused by the field h below θ_c is simply the average size of clusters, so that

$$\chi = \int_0^{s_{co}} sD(s)ds = (\theta_c - \theta)^{(3-\tau)/\sigma}, \tag{3a}$$

i.e., $\gamma = (3-\tau)/\sigma$.

The definitions (2c) and (2d), in combination with the relation $s_{co} \approx \xi^D$, give

$$D = 1/\sigma\nu. \tag{3b}$$

Near a second-order transition one can think of the order parameter as originating from clusters larger than a certain size, leading to scaling relations $\delta = 1/(\tau - 2)$ and $\beta = (\tau - 2)/\sigma$. It is not clear whether or not these relations can be expected to be valid here, since the cluster picture is in question in the case when there is a spontaneous flow.

From Eq. (1) one can see that the power spectrum $S(\omega)$ of the system response to a small white-noise perturbation is the dynamical susceptibility $|\chi(\omega)|^2$. In order to derive laws for the dynamic susceptibility at criticality, we note that at nonzero frequency ω only clusters of linear size less than $l \approx \omega^{-1/z}$, i.e., of volume s less than $\omega^{-D/z}$, give a contribution. Hence

$$S(\omega) \approx |\chi(\omega)|^2 = \int_0^{\omega^{-D/z}} sD(s)ds = \omega^{-D(3-\tau)/z} = \omega^{-\gamma/\nu z} \equiv \omega^{-\phi}. \tag{4}$$

Thus, the "noise" exponent ϕ is a critical exponent which can be related to the more traditional critical exponents through a simple scaling relation,

$$\phi = D(3-\tau)/z = \gamma/\nu z. \tag{5}$$

The scaling relation connecting the distribution of cluster sizes and the distribution of lifetimes derived in Ref. 3 [Eq. (2)] is, in fact, the scaling relation above in disguise.

The requirement of stationarity at the critical point imposes restrictions on the cluster distribution function. For a finite system of size L, addition of one unit "slope" is equivalent to adding a stream of "sand" of length of order L. So at the stationary state it must be followed in average by $\sim L^2$ sliding events. Another way to see this is to note that the dynamics on average is diffusionlike. It takes $\sim L^2$ steps for an extra unit to diffuse away. We expect clusters of linear size limited by L, i.e., of volume limited by L^D, so that the stationarity condition becomes

$$\int_0^{L^D} sD(s)ds = L^{D(3-\tau)} \approx L^2. \tag{6a}$$

Thus,

$$D(3-\tau) = 2 \text{ or } \gamma/\nu = 2. \tag{6b}$$

When inserted into Eq. (5) this gives a power spectrum $S(\omega) \approx \omega^{-2/z}$ which depends on the dynamical exponent z only. The exponent z is typically very close to the "mean field" or simple diffusion value 2. It is quite intriguing that this leads to a pure $1/f$ spectrum with exponent $\phi = 1$. [If the system is built by addition of particles rather than increase of the slope (model 1), each added particle must in average slide $\sim L$ steps, and so the exponent 2 should be replaced by 1 in (6a) and $\gamma/\nu = 1$, $\phi = 1/z$. Yet we have no idea what the "mean-field" value of z should be for this model.] While the argument leading to this conclusion may seem compelling, there is a hidden assumption that the distribution function $D(s)$ is independent of L for s smaller than the system ($s < L^D$). It could be (and is indeed the case for some models that we have investigated) that $D(s)$ has a prefactor which has a power-law dependence on L.

We mentioned in the introduction that the self-organized critical state is an attractor for the dynamical system. One relevant question is how fast a supercritical state would relax towards the critical state. Notice that the relaxation is caused by the flow j through the boundary. So we have the following differential equation:

$$d\epsilon/dt = c' L^{-1} j = -c L^{-1} \epsilon^{\beta}, \tag{7}$$

where $\epsilon \equiv (\theta - \theta_c)/\theta_c$ is a dimensionless parameter describing the "distance" to the attractor, c' and c are some constants. The factor L^{-1} on the right-hand side of (7) is due to the fact that ϵ is a bulk average quantity while the relaxation takes place only at the boundary. Note that only when $\beta = 1$ does the system approach the attractor exponentially, like most dynamical systems. In general, there is a power-law relaxation:

$$\epsilon(t)^{1-\beta} = \epsilon(0)^{1-\beta} - (1-\beta) c L^{-1} t, \tag{8}$$

so that the system reaches the critical state at a well defined time, $t = [\epsilon(0)^{1-\beta} L]/[c(1-\beta)]$.

Below the critical point the flow decays as

$$j(t) = \int_0^{s_{co}} j_s(t) D(s) ds, \tag{9}$$

where $j_s(t)$ is the response of clusters of size s. Assuming[8] a scaling function $j_s(t) \sim s^{1-z/D} g(t/s^{z/D})$, with $g(0) = g(1) = 0$ and $g(x) \sim x^{D/z-1}$ for $x \to 0$, we find

$$j(t) \sim t^{\phi-1} \exp[-(t/t_{co})^{D(\tau-2)/z}]. \tag{10}$$

Such stretched exponential relaxation has been observed in many glassy systems.

We have confirmed several of the power laws conjectured above by numerical calculations, and estimated the corresponding exponents. A typical numerical calculation is shown in Fig. 1. For the two-dimensional model, we get

$$\begin{gathered} \tau = 2.0, \quad \phi = 2 - b = 1.57, \quad D = 2.1, \\ \beta = 0.7, \quad \gamma = 1.35, \quad \nu = 0.74, \\ \sigma = 0.72, \quad z = 1.29; \end{gathered} \tag{11}$$

and for the three-dimensional model

$$\begin{gathered} \tau = 2.33, \quad \phi = 2 - b = 1.1, \quad D = 3.0, \\ \beta = 0.82, \quad \gamma = 1.7, \quad \nu = 0.85, \\ \sigma = 0.41, \quad z = 1.7. \end{gathered} \tag{12}$$

The accuracy of ν and σ is $\approx 10\%$; the accuracy of the other exponents is (2–5)%. These values obey the scaling relations 2, 4, and 5 above, within numerical accuracy. The relation $\beta = (\tau - 2)/\sigma$ is satisfied in 3D, but not in 2D. However, note that in 2D, $\tau \approx 2.0$ is a very special value, which could give rise to some logarithmic singularities. The hyperscaling $D = d - \beta/\nu$ does not apply here.

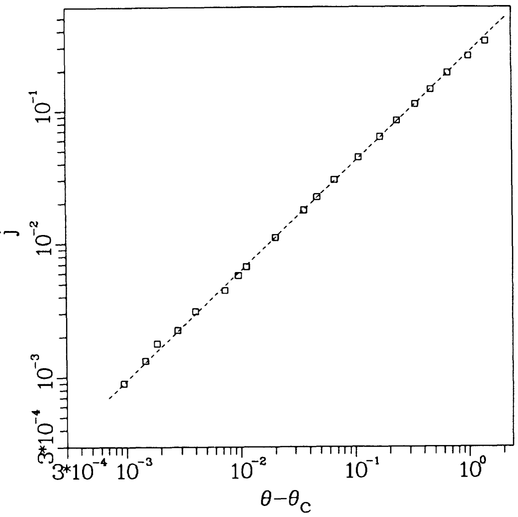

FIG. 1. Order parameter j vs $\theta - \theta_c$ for a 3D system of size $50 \times 50 \times 50$. The dashed line has a slope $\beta = 0.82$.

It would be desirable to have analytical tools to determine the exponents above, in order to check the scaling relations and to clarify the universality classes. There is no indication from Eqs. (11) and (12) that our model falls into any of the known universality classes. In fact, models with different symmetries are shown to have different exponents.[3,9] We have constructed a mean-field theory[10] for this model which gives $\beta = 1$, $\delta = 2$, $\gamma = 1$, and $z/\sigma D = 1$. When combined with Eqs. (3), (5), and (6), these mean-field exponents also imply that $\phi = 1$, $z = 2$, and $\nu = \frac{1}{2}$. The numerical values above are clearly different from the mean-field values. It would be interesting to investigate whether or not there exists an upper critical dimension above which mean-field theory is valid.

We urge that the phenomena described here be studied experimentally. Most importantly, one would like to verify the power laws in order to establish that we are indeed dealing with a critical phenomenon. The exponents may vary from system to system, but we expect the scaling relations to be more universal. Possible candidates are charge-density-wave systems, sliding vortex lattices in magnetic fields, sandpiles,[11] glassy systems, water flow, traffic, etc.

This work was supported by the Division of Materials Science, U.S. Department of Energy, under Contract No. DE-AC02-76CH00016, and by the Institute for Theoretical Physics, National Science Foundation Grant No. PHY82-17853 supplemented by NASA.

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[6]Here we use $-\tau+1$, instead of $-\tau$ as in Ref. 1, to be the exponent $D(s)$. This is more in tune with the usual definitions for critical phenomena.

[7]This corresponds to the study of a magnetic system at fixed magnetization rather than at fixed temperature.

[8]This is to say that $j_s(t)$ is self-similar. So one can write $j_s(t) \approx s^a g(t/t_s)$ where $t_s \approx s^{z/D}$ is the lifetime or relaxation time of clusters of size s. Then by definition $g(0)=g(1)=0$. The condition $\int_0^\infty j_s(t)dt=s$ sets $a=1-z/D$. The requirement that $j_s(t)$ should be independent of s for small enough t gives $g(x) \rightarrow x^{D/z-1}$, as $x \rightarrow 0$.

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