Chiral Gauge Theories on a Lattice

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We show that chiral gauge theories can be put on a lattice by exploiting the idea of adding extra degrees of freedom. We also examine the lattice chiral Schwinger model to find that the gauge interaction imposes a restriction on the values of a parameter peculiar to the lattice fermion.

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Quantization of chiral gauge theories has been a long-standing problem because of chiral anomalies which lead to gauge-noninvariant, nonunitary, and nonrenormalizable theories. The cancellation of the anomalies among the fermion contents has been a guiding principle of the building of unification models. However, recently the possibility of consistent quantization of the anomalous theories has been suggested by many authors. Jackiw and Rajaraman[1] show that the chiral Schwinger model, which is exactly solvable, yields a consistent and unitary but not gauge-invariant theory with a massive gauge boson. On the other hand, Faddeev[2] suggests that it may be possible to construct a gauge-invariant consistent theory by introducing extra bosons through the Wess-Zumino (WZ) term. Along this line, the chiral Schwinger model is reanalyzed to be understood that the effective action of Jackiw-Rajaraman coincides in some gauge with that computed in the theory with the WZ term.[3,4] Although attempts are made to construct consistent theories in four dimensions,[5] most of those are limited to two-dimensional theories.

These lessons on quantizing anomalous gauge theories tell us, "Any gauge theory must be constructed in a gauge-invariant manner. If not, add the extra freedom to recover the gauge invariance." Following this spirit one could put chiral gauge theories on a lattice. If this is the case one has a new possibility of lattice gauge theories, since in spite of the success in QCD,[6] those cannot be accommodated to the chiral gauge theories such as the Weinberg-Salam model because of the existence of a chiral-symmetry-breaking term called the Wilson term. Needless to say, the advantage of the lattice theory is that it retains the explicit gauge invariance even at quantum level and allows nonperturbative studies.

In this Letter we present a general prescription to construct a model of chiral gauge theory on a lattice which contains extra scalars to respect the chiral gauge invariance. Further employing the chiral Schwinger model, we calculate the effective action of the gauge field and show that the gauge boson becomes massive as in the continuum theory so that a parameter peculiar to the lattice fermion is restricted because of the gauge interaction. If mass generation occurs also in four dimensions, we can formulate $SU(2)_L \otimes U(1)$ electroweak theory on a lattice[7] which might alter the aspects of the Weinberg-Salam model such as particle contents and phase structure.

We start from the Wilson's lattice fermion action in D dimensions,[8]

$$S_f = -\frac{1}{2}\sum_{n,\mu}\bar{\psi}(n)\gamma_\mu[\psi(n+\mu)-\psi(n-\mu)] + \frac{1}{2}r\sum_{n,\mu}\bar{\psi}(n)[\psi(n+\mu)+\psi(n-\mu)-2\psi(n)]. \tag{1}$$

The second term on the right-hand side is the Wilson term, which has been introduced to avoid the species doubling and whose coefficient, r, is a free parameter[9,10] with $r>0$ to satisfy causality. It is also argued that a rigorous treatment of the fermion path integral by means of the coherent states requires $r=1$.[11] Our convention is that all the lattice quantities are dimensionless: $\psi \to a^{(1-D)/2}\psi$, $A_\mu \to a^{(2-D)/2}A_\mu$, $e \to a^{D/2-2}e$, where the left-hand sides denote those of the continuum and a is the lattice constant. Now let us build up a fermion action invariant under the chiral gauge transformation G_L,

$$\begin{aligned}\psi(n) &\to \psi^h(n) = [h(n)P_L + P_R]\psi(n),\\ \bar{\psi}(n) &\to \bar{\psi}^h(n) = \bar{\psi}(n)[h^\dagger(n)P_R + P_L],\end{aligned} \tag{2}$$

where $h(n)$ is an element of the compact Lie group G_L and $P_{L,R}=(1\pm\gamma_5)/2$. [For example, $\gamma_5 = i\gamma_1\gamma_2$ $(\gamma_1\gamma_2\gamma_3\gamma_4)$ in two (four) dimensions.] To this end we utilize the previous principle of keeping the chiral gauge invariance as well as the Euclidean (Lorentz) invariance. The first term on the right-hand side of (1), called the Dirac term, can be made invariant under (2) by the introduction of the link variable $U_\mu(n)=\exp[ieA_\mu(n+\mu/2)]$ transforming as

$$U_\mu(n) \to U_\mu^h(n) = h(n)U_\mu(n)h^\dagger(n+\mu). \tag{3a}$$

However, the Wilson term, which breaks the chiral invariance explicitly, cannot be rendered invariant simply by the introduction of $U_\mu(n)$. This violation of the chiral symmetry is the origin of the anomaly on a lattice.[9,12] In order to make it invariant, we must introduce a unitary field $g(n)=\exp[i\theta(n)]$ which compensates the change of the fermions under (2) by transforming as

$$g(n)\rightarrow g(n)h^{\dagger}(n). \tag{3b}$$

Furthermore, if we require locality and Hermiticity, the following gauge-invariant action is obtained almost uniquely[13]:

$$\begin{aligned} S_f[\psi,\bar\psi,A,\theta] &= S_D[\psi,\bar\psi,A]+S_W[\psi,\bar\psi,\theta],\\ S_D[\psi,\bar\psi,A] &= -\tfrac{1}{2}\sum_{n,\mu}\{\bar\psi(n)\gamma_\mu[U_\mu(n)P_L+P_R]\psi(n+\mu)-\bar\psi(n+\mu)\gamma_\mu[U_\mu^\dagger(n)P_L+P_R]\psi(n)\},\\ S_W[\psi,\bar\psi,\theta] &= \frac{r}{2}\sum_{n,\mu}\{\bar\psi(n)[g^\dagger(n)P_R+P_L][g(n+\mu)P_L+P_R]\psi(n+\mu)\\ &\quad+\bar\psi(n+\mu)[g^\dagger(n+\mu)P_R+P_L][g(n)P_L+P_R]\psi(n)-2\bar\psi(n)[g^\dagger(n)P_R+P_L][g(n)P_L+P_R]\psi(n)\}. \end{aligned} \tag{4}$$

Here the situation of the chiral gauge invariance should be compared to the continuum theory where the addition of the WZ term kills the anomaly from the fermionic functional measure, while in the lattice theory the introduction of $g(n)$ wipes out the chiral-breaking nature of the Wilson term. In this sense the lattice action (4) might be regarded as an alternative to the classical action plus the WZ term in the continuum theory. Since the WZ terms have involved form, our action could be more tractable when one considers the extension to realistic models.[14]

Adopting the simplest example U(1) as the gauge group, we now illustrate the calculation of the effective action of the gauge field defined by

$$\exp\{W[A]\}=\int[d\theta]\exp\{S_g[U]+\Gamma[A,\theta]\}, \tag{5}$$

where

$$\Gamma[A,\theta]=\ln\int[d\psi\, d\bar\psi]\exp\{S_f[\psi,\bar\psi,A,\theta]\}, \tag{6}$$

with S_f from (4). Because of the invariance of the fermion measure, $\Gamma[A,\theta]$ can be rewritten as follows:

$$\begin{aligned} \Gamma[A,\theta] &= \ln\int[d\psi^{g^{-1}}d\bar\psi^{g^{-1}}]\exp\{S_D[\psi^{g^{-1}},\bar\psi^{g^{-1}},A]+S_W[\psi^{g^{-1}},\bar\psi^{g^{-1}},\theta]\}\\ &= \ln\int[d\psi\, d\bar\psi]\exp\{S_D[\psi,\bar\psi,A-\nabla\theta/e]+S_W[\psi,\bar\psi,0]\}\\ &= \Gamma[A-\nabla\theta/e,0], \end{aligned} \tag{7}$$

where

$$\nabla_\mu\theta(n)=\theta(n+\mu)-\theta(n). \tag{8}$$

The fact that $\Gamma[A,\theta]$ is a function of $A_\mu-\nabla_\mu\theta/e$ tells us that it is gauge invariant and that θ can be regarded as the Goldstone-Higgs field. $\Gamma[A,0]$ can be worked out perturbatively:

$$\Gamma[A,0]=\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{n_1,\mu_1}\cdots\sum_{n_k,\mu_k}A_{\mu_1}(n_1+\mu_1/2)\cdots A_{\mu_k}(n_k+\mu_k/2)\Gamma_{\mu_1\cdots\mu_k}(n_1+\mu_1/2,\ldots,n_k+\mu_k/2), \tag{9}$$

where $\Gamma_{\mu_1\mu_2\cdots\mu_k}$ corresponds to the k-point vertex function in the continuum theory and is given by

$$\Gamma_{\mu_1\mu_2\cdots\mu_k}(n_1+\mu_1/2,\ldots,n_k+\mu_k/2)=-\sum_{N=1}^{\infty}\frac{1}{N!}\left.\frac{\partial^k \mathrm{Tr}[(iS\times i\Sigma)^N]}{\partial A_{\mu_1}(n_1+\mu_1/2)\cdots\partial A_{\mu_k}(n_k+\mu_k/2)}\right|_{A=0}. \tag{10}$$

Here S and Σ are the free propagator of the fermion and the gauge-fermion vertex, respectively.

Now we apply these prescriptions to the chiral Schwinger model. The detailed calculation of $\Gamma_{\mu_1\mu_2\cdots}$ will be presented elsewhere.[15] Here we write down only the result in the continuum ($a\rightarrow 0$) limit:

$$\Gamma[A,0]=-\frac{e^2}{8\pi}\int d^2x\, A_\mu(x)\left[\alpha(r)\delta_{\mu\nu}-(\delta_{\mu\alpha}+i\epsilon_{\mu\alpha})\frac{\partial_\alpha\partial_\beta}{\Box}(\delta_{\nu\beta}+i\epsilon_{\nu\beta})\right]A_\nu(x), \tag{11}$$

where the higher-order terms in A_μ vanish in the continuum limit[16] and $\alpha(r)$ is a dimensionless constant defined by

$$\alpha(r)=1+\frac{r^2}{\pi}\int_{-\pi}^{\pi}d^2q\,\frac{C(q)^2\cos^2q_1-2C(q)\cos q_1\sin^2q_1}{[\sin^2q_1+\sin^2q_2+r^2C(q)^2]^2}, \tag{12}$$

TABLE I. Numerical values of $\alpha(r)-1$ and $\beta(r)$ for $r=0$, 0.1, 1, r_c, 2, and ∞, which are the integrals appearing in the A^2 term of the fermion determinant and in the anomaly, respectively. The analytical calculation tells us that $\beta(r)=-1$ for all r, but only that $\alpha(0)=4$ and $\alpha(\infty)=0$.

r	$\alpha(r)-1$	$\beta(r)$
0	3	−1
0.1	2.898 611 88 ± 0.000 001 41	− 1.000 000 39 ± 0.000 005 52
1.0	0.508 883 10 ± 0.000 000 01	− 0.999 999 99 ± 0.000 000 01
1.518 431 33	0.000 000 00 ± 0.000 000 02	− 1.000 000 00 ± 0.000 000 02
2.0	− 0.261 710 14 ± 0.000 000 03	− 1.000 000 00 ± 0.000 000 02
∞	−1	−1

with $C(q)=\sum_{\mu=1,2}(1-\cos q_\mu)$. In contrast to the continuum theory, in which the A^2 term is introduced with an arbitrary coefficient by one's taking account of the violation of the gauge symmetry due to the anomaly,[1,17] we do have the A^2 term as a consequence of the lattice calculation. This term originates from the mixing of the left- and right-handed fermions through the Wilson term. $\alpha(r)$ seems to be a monotonically decreasing function of r by numerical analysis and its values for several r are listed in Table I. To demonstrate the reliability of the numerical calculation, we also compute the coefficient of the chiral anomaly,[18] $\beta(r)$. It has almost similar form to $\alpha(r)-1$ as given by (12) and is defined by

$$\langle\partial_\mu J_{L\mu}\rangle=-i\beta(r)(e^2/4\pi)\epsilon_{\mu\nu}\partial_\mu A_\nu, \tag{13}$$

and

$$\beta(r)=\frac{r^2}{\pi}\int_{-\pi}^{\pi}d^2q\frac{C(q)^2\cos q_1\cos q_2-2C(q)\cos q_1\sin^2 q_2}{[\sin^2 q_1+\sin^2 q_2+r^2C(q)^2]^2}, \tag{14}$$

which can be analytically evaluated to be -1 for all r.[12] Thus we have

$$\Gamma[A,\theta]=-\frac{e^2}{8\pi}\int d^2x[A_\mu(x)-e^{-1}\partial_\mu\theta(x)]\left[\alpha(r)\delta_{\mu\nu}-(\delta_{\mu\alpha}+i\epsilon_{\mu\alpha})\frac{\partial_\alpha\partial_\beta}{\Box}(\delta_{\nu\beta}+i\epsilon_{\nu\beta})\right][A_\nu(x)-e^{-1}\partial_\nu\theta(x)]$$
$$=\Gamma[A,0]+\Gamma_{WZ}[A,\theta], \tag{15}$$

where Γ_{WZ} denotes the WZ term, which indicates that θ also plays the role of the WZ scalar. After performing the θ integral by regarding it as a Gaussian integral,[19] provided $\alpha(r)>1$, we obtain

$$W[A]=\tfrac{1}{2}\int d^2x\,A_\mu(x)\left[\Box-\frac{e^2}{4\pi}\frac{\alpha(r)^2}{\alpha(r)-1}\right]\left(\delta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\Box}\right)A_\nu(x). \tag{16}$$

This shows that if $\alpha(r)>1$ the resulting theory is unitary and then the transverse component of A_μ has a mass m such that

$$m^2=\frac{e^2}{4\pi}\frac{\alpha(r)^2}{\alpha(r)-1}, \tag{17}$$

in agreement with previous works.[1,4,17] Instead of the θ integration we find the same result by use of bosonization. For $\alpha(r)=1$ the mass diverges and the massive gauge boson disappears.[1,20]

The condition $\alpha(r)\geq 1$ imposes a restriction on r such that $0<r\leq r_c$ in contrast to the free theory where $0<r<\infty$. Here r_c is defined through $\alpha(r_c)=1$, whose numerical value is about 1.5184313. m^2 takes a value between e^2/π and infinity for $0<r\leq r_c$.

To summarize, we have developed the idea of quantizing chiral gauge theories and found that the fundamental action of the lattice fermions contains extra scalars necessary for the action to be invariant under the chiral gauge transformation. We have then discussed the effective action of gauge fields in the lattice chiral Schwinger model. The resultant theory is unitary for some range of the parameter r, the coefficient of the Wilson term, to wit, the gauge interaction places a restriction on the range of it. However, it should be emphasized that we are indebted to the exact solvability of the model in two dimensions for reaching these consequences.

As for non-Abelian theories in four dimensions, $\Gamma[A,\theta]$ of Eq. (7) is a function of $A_\mu^g=gA_\mu g^\dagger-(i/e)g\,\partial_\mu g^\dagger$ instead of $A_\mu-(1/e)\partial_\mu\theta$. In general, however, $\Gamma[A,\theta]$ cannot be calculated exactly, and so we must adopt some approximation methods. With the aid of the weak-coupling expansion $\Gamma[A,\theta]$ is shown to possess an A^2 term, which might indicate the mass of the gauge field. In contrast to the two-dimensional case, the limit $a \rightarrow 0$ cannot be taken, which causes ambiguities in the theory due to the counterterms: The problems of renormalizability and unitarity might be subtle in this framework. Nevertheless, our theory has the advantage of allowing nonperturbative analyses such as the strong-coupling expansion and the Monte Carlo simulations, since it is defined on a lattice from scratch. These subjects are very intriguing because our theory might offer a different phase structure from the standard Weinberg-Salam theory.

After the submission of this work we have received a report by Smit.[14] We would like to thank him for sending it to us.

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