Chiral Gauge Theories on a Lattice

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A chiral gauge-invariant lattice fermion action is derived from a chiral gauge-variant Wilson fermion action without change in its partition function. By use of this action it is shown that anomalous gauge theories in four dimensions are renormalizable. An application to the chiral Schwinger model shows that it is unitary for a range of Wilson parameter r.

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Nowadays, chiral gauge theories play important roles in elementary-particle physics. In particular, it is very important to analyze chiral gauge theories with the lattice regularization. Unfortunately, however, under rather mild conditions it was proven that chiral fermions cannot be defined on a lattice without species doubling.[1] Wilson avoided this problem by introducing an extra term which lifts the mass of the doubling modes.[2] This term, however, explicitly breaks the chiral invariance of the action. In the case of the chiral gauge theory, this means that the Wilson fermion breaks not only chiral symmetry but also gauge invariance. The lack of manifest gauge invariance is particularly undesirable because that represents perhaps the most important feature of standard lattice gauge theory.

Can we overcome this difficulty? The answer is probably no. The reason is as follows. In the usual perturbation theory it is well known that the chiral gauge theory has the (local) non-Abelian anomaly.[3] Since the lattice theory is a well-defined, regularized theory, anomalous symmetry cannot be maintained on the lattice. Therefore it is expected that chiral gauge symmetry must be broken on a lattice. Indeed, the Wilson term gives the correct non-Abelian anomaly in the perturbative continuum limit.[4] Furthermore, the explicit breakdown of the chiral gauge symmetry is not unique to the lattice regularization: Pauli-Villars regularization or dimensional regularization, for example, also breaks the chiral gauge symmetry. When the cutoff is finite the chiral gauge symmetry is broken not only for the anomalous case but also for the anomaly-free case such as SU(2). In this sense the chiral gauge symmetry is not a true symmetry of regulated *quantum* field theory. From the above consideration, it is unavoidable to use a chiral gauge-variant action, such as the Wilson fermion action, at the starting point. However, it is uncomfortable to use a gauge-variant action. So in this Letter, starting with the Wilson fermion action, I show that it is possible to make it *gauge invariant* by introducing an auxiliary field, without changing the partition function. After integration of this auxiliary field I get a gauge-invariant but nonlinear fermionic action for chiral gauge theories on a lattice. There are two different purposes to the use of the lattice action for chiral gauge theories: One is, of course, to analyze the nonperturbative property of electroweak interactions on a lattice. The other aim is to answer whether anomalous gauge theories are consistent or not. In this Letter I discuss only renormalizability and find that anomalous gauge theories in 2D and 4D are renormalizable. Finally, I apply my method to the chiral Schwinger model[5] and discuss the unitarity of the theory. The detailed calculations of this Letter will be published elsewhere.[6]

The action of the chiral gauge theory with Wilson fermions is given by

$$S=S_0+S_W+S_G, \tag{1}$$

where

$$S_0=\frac{a^d}{2a}\sum_{\mathbf{n},\mu}\bar{\psi}_\mathbf{n}\gamma_\mu[(L_{\mathbf{n},\mu}P_L+R_{\mathbf{n},\mu}P_R)\psi_{\mathbf{n}+\hat{\mu}}-(L^\dagger_{\mathbf{n}-\hat{\mu},\mu}P_L+R^\dagger_{\mathbf{n}-\hat{\mu},\mu}P_R)\psi_{\mathbf{n}-\hat{\mu}}],$$

$$S_W=-\frac{ra^d}{2a}\sum_{\mathbf{n},\mu}\bar{\psi}_\mathbf{n}(\psi_{\mathbf{n}+\hat{\mu}}+\psi_{\mathbf{n}-\hat{\mu}}-2\psi_\mathbf{n})+Ma^d\sum_{\mathbf{n},\mu}\bar{\psi}_\mathbf{n}\psi_\mathbf{n}.$$

Here $P_{L,R}=(1\pm\gamma_5)/2$, $L_{n,\mu}$ is a left-handed (right-handed) gauge field, S_0 is a gauge-invariant part of the fermion action, and S_W is the Wilson plus mass term for the fermion. $\mathbf{n}$ is the position of a lattice site, and $\hat{\mu}$ is a vector whose μth component is 1 and all others 0. For generality I add the fermion mass term to the action though it is usually prohibited by gauge invariance. S_G is a pure gauge action for $L_{\mathbf{n},\mu}$ an $R_{\mathbf{n},\mu}$. It is noted that S_W is gauge *variant* and therefore both left- and right-handed fermions are necessary. Of course, they can have different quantum numbers.

Next I rewrite the partition function Z:

$$Z=\int\prod_{\mathbf{n}} dg_{\mathbf{n}}^L\, dg_{\mathbf{n}}^R\, \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\,\mathcal{D}L\,\mathcal{D}R\, e^{S(g)},$$

where

$$S(g)=S_0+S_G-\frac{ra^d}{2a}\sum_{\mathbf{n},\mu}\bar{\psi}_{\mathbf{n}}[g_{\mathbf{n}}^{L\dagger}P_R(g_{\mathbf{n}+\hat{\mu}}^R\psi_{\mathbf{n}+\mu}+g_{\mathbf{n}-\hat{\mu}}^R\psi_{\mathbf{n}-\hat{\mu}}-2g_{\mathbf{n}}^R\psi_{\mathbf{n}})$$
$$+g_{\mathbf{n}}^{R\dagger}P_L(g_{\mathbf{n}+\hat{\mu}}^L\psi_{\mathbf{n}+\hat{\mu}}+g_{\mathbf{n}-\hat{\mu}}^L\psi_{\mathbf{n}-\hat{\mu}}-2g_{\mathbf{n}}^L\psi_{\mathbf{n}})]+Ma^d\sum_{\mathbf{n}}\bar{\psi}_{\mathbf{n}}(g_{\mathbf{n}}^{L\dagger}g_{\mathbf{n}}^R P_R+g_{\mathbf{n}}^{R\dagger}g_{\mathbf{n}}^L P_L)\psi_{\mathbf{n}}.$$

Here $g_n^{L,R}$ is a matrix-valued scalar field. To get the above equation, inserting the identity that $\int dg_{\mathbf{n}}^{L,R}=1$ where $dg_{\mathbf{n}}^{L,R}$ is the Haar measure on the gauge group, I make a change of integration variables:

$$\psi_{\mathbf{n}}=(g_{\mathbf{n}}^L P_L+g_{\mathbf{n}}^R P_R)\psi_{\mathbf{n}}',\quad L_{\mathbf{n},\mu}=g_{\mathbf{n}}^L L_{\mathbf{n},\mu}' g_{\mathbf{n}+\hat{\mu}}^{L\dagger},$$
$$\bar{\psi}_{\mathbf{n}}=\bar{\psi}_{\mathbf{n}}'(g_{\mathbf{n}}^{L\dagger}P_R+g_{\mathbf{n}}^{R\dagger}P_L),\quad R_{\mathbf{n},\mu}=g_{\mathbf{n}}^R R_{\mathbf{n},\mu}' g_{\mathbf{n}+\hat{\mu}}^{R\dagger}.$$

I omit the primes on the new fields in $S(g)$.

An interesting property of $S(g)$ is that it now has a *gauge invariance:* $S(g)$ is invariant under the gauge transformation

$$\psi_{\mathbf{n}}'=(h_{\mathbf{n}}^L P_L+h_{\mathbf{n}}^R P_R)\psi_{\mathbf{n}},\quad L_{\mathbf{n},\mu}'=h_{\mathbf{n}}^L L_{\mathbf{n},\mu}h_{\mathbf{n}+\hat{\mu}}^{L\dagger},\quad g_{\mathbf{n}}^{L\prime}=g_{\mathbf{n}}^L h_{\mathbf{n}}^{L\dagger},$$
$$\bar{\psi}_{\mathbf{n}}'=\bar{\psi}_{\mathbf{n}}(h_{\mathbf{n}}^{L\dagger}P_R+h_{\mathbf{n}}^{R\dagger}P_L),\quad R_{\mathbf{n},\mu}'=h_{\mathbf{n}}^R R_{\mathbf{n},\mu}h_{\mathbf{n}+\hat{\mu}}^{R\dagger},\quad g_{\mathbf{n}}^{R\prime}=g_{\mathbf{n}}^R h_{\mathbf{n}}^{R\dagger},$$

where $h_{\mathbf{n}}$ is a gauge-transformation function. Some years ago it was pointed out that formal gauge invariance can generally be derived from gauge-variant theory.[7] The above result is an application of that method.

There are some remarks.

(1) It is not substantially more difficult to do *Monte Carlo* simulations by use of the action $S(g)$ rather than S.

(2) The integration over $g_{\mathbf{n}}^{L,R}$ has the same effect as an integration over "all" gauge transformations. This method is proposed by Harada and Tsutsui to quantize the anomalous gauge theory in the continuum approach.[8]

(3) After integrating the fermion field, I get the following partition function:

$$Z=\int\prod_{\mathbf{n}} dg_{\mathbf{n}}^L\, dg_{\mathbf{n}}^R\,\mathcal{D}L\,\mathcal{D}R\exp[S^{\mathrm{WZ}}(g,L,R)+S_{\mathrm{eff}}(1,L,R)],\tag{2}$$

where I define

$$S_{\mathrm{eff}}(g,L,R)\equiv\ln\left[\int\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\, e^{S(g)}\right],$$

$$S^{\mathrm{WZ}}(g,L,R)\equiv S_{\mathrm{eff}}(g,L,R)-S_{\mathrm{eff}}(1,L,R).$$

S^{WZ} is nothing but the Wess-Zumino term.[9] This formula (2) shows that I integrate the Wess-Zumino term over $g_{\mathbf{n}}^{L,R}$ to calculate Z.[8]

(4) If I add a gauge-invariant kinetic term for $g_{\mathbf{n}}^{L,R}$ to the original action S, I can identify $g_{\mathbf{n}}^{L,R}$ as the angular part of a fundamental Higgs field. Under this identification, the model is equivalent to the one proposed by several authors.[10] However, I do not have to add the kinetic term in the present approach and, furthermore, the theory becomes unrenormalizable in 4D if I add the kinetic term.

In some cases I can perform the g integral explicitly unless the kinetic term exists. Some results for when only the left-handed gauge field exists are listed below. First I define

$$K\equiv\int\prod_{\mathbf{n}} dg_{\mathbf{n}}^L\exp[S(g)-S_0-S_G].$$

(1) U(1) case: The number of left-handed fermions is k and U(1) charges are $q_1,q_2,\ldots,q_k$. Then

$$K=\sum_{s=0}^{\infty}\sum_{\sum l_j+m_j=s}\ \sum_{\sum q_j(l_j-m_j)=0}\ \prod_j\frac{1}{l_j!m_j!}(\bar{A}_{\mathbf{n}}^j)^{l_j}(A_{\mathbf{n}}^j)^{m_j},$$

where

$$A_{\mathbf{n}}^j=a^d\bar{\psi}_{\mathbf{n}}^j P_R\left[M-\tfrac{1}{2}ra\textstyle\sum_\mu\nabla_\mu^2\right]\psi_{\mathbf{n}}^j,$$

$$\bar{A}_{\mathbf{n}}^j=a^d\left[M-\tfrac{1}{2}ra\textstyle\sum_\mu\nabla_\mu^2\right]\bar{\psi}_{\mathbf{n}}^j P_L\psi_{\mathbf{n}}^j.$$

For example,

$$K=\prod_n I_0(2(\bar{A}_{\mathbf{n}}A_{\mathbf{n}})^{1/2}),\quad\text{for } k=1,$$

where I_0 is the zeroth-order modified Bessel function.

(2) SU(2) case: A left-handed fermion is the doublet and a right-handed fermion is the singlet. Then

$$K=\prod_n I_1(2\sqrt{J_{\mathbf{n}}})/\sqrt{J_{\mathbf{n}}},$$

where

$$J_{\mathbf{n}} = \mathrm{Tr}(\bar{A}_{\mathbf{n}} A_{\mathbf{n}}) + \det(\bar{A}_{\mathbf{n}} + A_{\mathbf{n}})$$

and I_1 is the first-order modified Bessel function.

The above results are *gauge invariant, local* but *not bilinear* in the fermion fields.

Now I briefly discuss the renormalizability for the nonlinear action in lattice perturbation theory by following the method of Kawai, Nakayama, and Seo.[11] I define E_G, E_g, and E_f as the numbers of external lines of gauge fields, ghosts, and fermions, respectively, V_n^G, V_n^{Gg}, and V_n^{Gf} as the numbers of gauge, gauge-host, and gauge-fermion n-point vertices, respectively, and $V_{n,k}^f$ as the number of $4n$-point fermion self-interactions, whose coefficient is M_k. In 2D, the superficial degree of divergence D of a Feynman diagram becomes

$$D = 2 - \tfrac{1}{2} E_f - \sum_{n=3}^{\infty} (n-2)[V_n^G + V_n^{Gg} + V_n^{Gf}] - \sum_{n=1}^{\infty} \sum_{k=0}^{2n} k V_{n,k}^f .$$

Because D decreases and becomes negative as V's increase, this anomalous chiral gauge theory is renormalizable in two dimensions. In four dimensions,

$$D = 4 - (E_G + E_g) - \tfrac{3}{2} E_f - \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} k V_{n,k}^f ,$$

and this shows that the anomalous gauge theory is also renormalizable in 4D.

In the remaining part of this Letter I show the perturbative result of an application to the chiral Schwinger model. Note that the nonperturbative effect, which is most important and difficult, is not treated here. I make this calculation to see that the method works well as a regularization for perturbation. An action of the chiral Schwinger model is equal to S in (1) with $L_{n,\mu} \in \mathrm{U}(1)$ and $R_{n,\mu} = 1$. After integrating out the fermion field I get

$$S_{\mathrm{eff}}(A,\theta) = -\sum_{\mathbf{n}} \Big[\frac{K+1}{8\pi} A_{\mathbf{n}}^{\mu} A_{\mathbf{n},\mu} + \frac{1}{4\pi} \partial^{\mu} A_{\mathbf{n},\mu} \frac{1}{\Delta} \partial^{\nu} A_{\mathbf{n},\nu} + \frac{1}{4\pi} \partial^{\mu} A_{\mathbf{n},\mu} \frac{1}{\Delta} \epsilon^{\alpha\beta} \partial_{\alpha} A_{\mathbf{n},\beta} + \frac{K-1}{4\pi} \theta_{\mathbf{n}} \partial^{\mu} A_{\mathbf{n},\mu} + \frac{K-1}{8\pi} \partial^{\mu} \theta_{\mathbf{n}} \partial_{\mu} \theta_{\mathbf{n}} - \frac{1}{4\pi} \theta_{\mathbf{n}} \epsilon^{\alpha\beta} \partial_{\alpha} A_{\mathbf{n},\beta} \Big] + S_G ,$$

where

$$\Delta = \partial^{\mu} \partial_{\mu}, \quad L_{\mathbf{n},\mu} = \exp[iaA_{\mathbf{n},\mu}], \quad g_{\mathbf{n}}^L = \exp[i\theta_{\mathbf{n}}], \quad \frac{K}{4\pi} = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{d^2 p}{(2\pi)^2} \left[\frac{\sin^2 p_{\mu}}{F(p)} + \frac{(2\sin^2 p_{\mu} - \sin^2 p)\cos^2 p_{\mu}}{F(p)^2} \right],$$

$$\sin^2 p = \textstyle\sum_{\mu} \sin^2 p_{\mu}, \quad F(p) = \sin^2 p + \left[r \textstyle\sum_{\mu} (1 - \cos p_{\mu}) \right]^2 .$$

It is noted that the above result is obtained by the fermion one-loop calculation; $A_{\mathbf{n},\mu}$ is treated as the external field. The nonperturbative effect of *lattice* U(1) gauge fields is not considered here. Then I integrate the θ_n field in the range $-\infty < \theta_n < \infty$ and get

$$S_{\mathrm{eff}}(A) = -\frac{1}{4e^2} \sum_{n,\mu \neq \nu} F^{\mu\nu}(n) \frac{1}{\Delta} (\Delta - m^2) F_{\mu\nu}(n), \quad (3)$$

where

$$m^2 = \frac{e^2}{4\pi} \frac{K^2}{K-1} .$$

It is noted that the result (3) is gauge invariant. The authors of Ref. 8 obtained the same result. The renormalization ambiguity K of Ref. 5 (denoted as a there) now can be determined by the action $S(g)$. By the numerical integration of K, I find that

$$m^2 > 0, \quad \text{for } 0 < r^2 \leq r_c^2 ,$$

where $r_c \simeq 1.5$. This shows that the theory has unitarity in this range of r as well as gauge invariance: The chiral Schwinger model is renormalizable, gauge invariant, and unitary in this range of Wilson parameter r. The condition that $0 < r^2 \leq r_c^2$ is consistent with the condition that $0 < r^2 \leq 1$, which is implied by the reflection positivity,[12] and thus it shows that the reflection positivity is a *sufficient* condition for unitarity of theories.

If I add a gauge-invariant kinetic term for g_n^L,

$$S_{\mathrm{H}} = \frac{H}{a^2} \sum_{\mathbf{n},\mu} [g_{\mathbf{n}}^L L_{\mathbf{n},\mu} g_{\mathbf{n}+\hat{\mu}}^{L\dagger} + g_{\mathbf{n}+\hat{\mu}}^L L_{\mathbf{n},\mu}^{\dagger} g_{\mathbf{n}}^{L\dagger}],$$

I can identify g_n^L as a Higgs field[10] and get Eq. (3) with

$$m^2 = \frac{e^2}{4\pi} \frac{(K+2H)^2}{K+2H-1}$$

after θ integration. This shows that the addition of S_{H} changes the mass for the gauge field, but in the present case the difference can be absorbed into the renormalization ambiguity of Ref. 5.

Finally, I discuss the remaining problems. Since my calculation for the chiral Schwinger model is only a perturbative one, we should study the nonperturbative effect by the strong-coupling expansion or the numerical method. Furthermore, we should investigate theories in 4D since they are physically most interesting. It is, of course, more difficult to analyze the theory in 4D even if we have a good definition on a lattice. The most important and difficult point is to investigate the continuum limit. For example, since there is no apparent difference between anomalous and anomaly-free theories in my formalism, the difference between them should appear in their continuum limits. There are several possibilities: Anomalous gauge theory may give a mass to the gauge field even in 4D, as the chiral Schwinger model does, or it may become trivial. The effect of the Wilson term may be nontrivial in the continuum limit even for the anomaly-free case, perhaps even inducing a mass for the gauge boson.

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