Nonlinear Ion-Temperature-Gradient–Driven Instability in Low-Collisionality Plasmas

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A novel theory for the self-consistent evolution of a nonlinear trapped-ion temperature-gradient–driven instability, based on the turbulent trapping of resonant ions in the electrostatic potential of the waves, is proposed. Threshold-dependent, non-steady-state turbulence (nonlinear instability) is shown to develop. The resulting anomalous thermal and particle transports act to reconfigure the equilibrium temperature and density profiles in such a way as to return the system towards marginality. Implications of the theory for present and future-generation fusion experiments are discussed.

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As present and future-generation toroidal-fusion experimental devices approach very low-collisionality regimes (i.e., $\nu_{*e},\nu_{*i} \ll 1$, where ν_* is the ratio of the effective collision frequency to the trapped-particle bounce frequency), a class of instabilities associated with ions trapped in the magnetic field inhomogeneity[1,2] are theoretically predicted to set in and considerably impair confinement. While the linear theory of these modes has been developed in considerable detail, little if any treatment of their nonlinear evolution and saturation has been proposed. In this Letter we report discovery of a novel relaxation mechanism for the trapped-ion temperature-gradient–driven mode,[1,2] which gives rise to a nonlinear (i.e., finite-amplitude) instability with a threshold on the value of $\eta_i = d\ln T_i/d\ln n_i$ (T_i and n_i are respectively the ion temperature and particle density). The relaxation is basically a Fokker-Planck process whereby the diffusion of a phase-space "eddy" of trapped ions in the turbulent bath of fluctuations is compensated for by a drag, induced by the polarization of the dielectric medium. If this assemblage of ions remains trapped in the electrostatic wave for a time (τ_{tr}) exceeding the typical autocorrelation time of the background turbulent spectrum (τ_c), then large anomalous transport will ensue which, by modifying equilibrium profiles, acts to return η_i to its marginal value.

The mode under consideration here[1,2] is triggered by unfavorable trapped-ion drifts (i.e., $\bar{\omega}_{di}\omega_{*i} > 0$, where $\bar{\omega}_{di}$ is the bounce-averaged magnetic precessional drift of the trapped ions, and ω_{*i} is the diamagnetic-drift frequency), and propagates in the *ion*-drift direction, thus allowing the possibility for resonant interaction between wave and particle ($\omega \simeq \bar{\omega}_{di}$). In the latter respect, as well as the class of particles driving it, the mode is fundamentally different from the more conventional (fluid-like) toroidal η_i mode.[3] We undertake here to investigate the nonlinear dynamics of the trapped-ion mode from the perspective of *turbulent* detrapping of resonant ions from the electrostatic potential of the wave. Our approach is thus in marked contrast to an earlier work[4] which explored the nonlinear saturation of the mode due to *coherent* trapping of the resonant ions. While the idea of coherent trapping is an interesting one in its own right, it is unlikely to be a faithful description of what actually occurs in practice. Principally, this is because the width of the modes is comparable to the spacing between them, so that nonlinear particle orbits can more realistically be expected to be stochastic rather than spatially organized. In view of the intractability of following exact particle trajectories, we adopt instead a statistical approach and consider the self-consistent evolution of the two-point correlation function of the trapped-particle distribution function in phase space. Electron collisions are the principal dissipation process here. The other potential dissipation mechanism, namely, Landau damping on the nonadiabatically responding circulating ions, is negligible since $\delta h_{\mathrm{i,circ}} = O(\bar{\omega}_{di}/\omega_{ti}) \ll 1$, where δh is the nonadiabatic part of the ion distribution function, and $\omega_{ti} = v_{ti}/Rq$ is the transit frequency. Electrons are taken to be sufficiently collisional (i.e., their effective collision frequency, $\nu_{\mathrm{eff},e}$, greater than all other frequencies of interest) to destroy strong, short-scale phase-space correlation on a rapid time scale. Their response is therefore laminar, and we focus attention on the nonlinear evolution of trapped ions.

We begin with the equation that describes the temporal evolution of the ensemble-averaged two-point correlation of the trapped-ion distribution function in phase space. This is obtained upon multiplication of the bounce-averaged nonlinear drift kinetic equation[5] for $\delta h(1)$ (where δh is the nonadiabatic part of the ion distribution function, and the argument "1" denotes a given point in phase space) by $\delta h(2)$, ensemble averaging, and symmetrizing the result. One then arrives at

$$[\partial/\partial t + \bar{v}_{di}(\bar{E}_1\,\partial/\partial y_1 + \bar{E}_2\,\partial/\partial y_2) + \mathcal{T}]\langle \delta h(1)\delta h(2)\rangle = \mathcal{S}, \qquad (1)$$

where $\langle \cdots \rangle$ denotes an ensemble average,

$$\mathcal{T}\langle \delta h(1)\delta h(2)\rangle = \frac{\pi c}{B}\sum_{m'}\sum_{k,k',k'',\omega,\omega',\omega''} k'k''(2\pi m')\{\langle \exp[i(k''-k')y_1 - iky_2]\delta h^*_{k,\omega}(2) \times \overline{[\delta\phi^*_{k',\omega'}(1)]}\delta h_{k'',\omega''}(1)\rangle \exp[2\pi i m' n' q(1)]\hat{s}_1 + (1\rightarrow 2)\} \quad (2)$$

is the nonlinear $\mathbf{E}\times\mathbf{B}$ triple correlation with $k''=k'+k$ understood [the notation $(1\rightarrow 2)$ stands for a second set of terms identical to the first, but with arguments interchanged and complex conjugated, as appropriate], and

$$\mathcal{S} = (e/T_i)\langle f(1)\rangle[\langle \delta h(2)\,\partial\overline{\delta\phi(1)}/\partial t\rangle + \langle \delta h(2)\,\partial\overline{\delta\phi(1)}/\partial y_1\rangle \bar{\omega}^T_{*i} + (1\rightarrow 2)] \quad (3)$$

is a source term which drives the evolution of the two-point correlation function. In Eq. (1),

$$\bar{\omega}_{di} = (q/r)\bar{v}_{di} = -k\rho_i v_{ti} G(\hat{s},\theta_0)/2R, \quad v_{ti} = (2T_i/m_i)^{1/2}, \quad \bar{E} = E/T_i,$$

$$\omega^T_{*i} = (q/r)\bar{\omega}^T_{*i} = \omega_{*i}[1+\eta_i(\bar{E}-\tfrac{3}{2})], \quad \omega_{*i} = -k\rho_i v_{ti}/2L_n, \quad L_n = -(d\ln n/dr)^{-1},$$

$G(\hat{s},\theta_0)$ is a function of the shear and the poloidal turning point, T_i is the ion temperature, q is the safety factor, $\hat{s}=rq'/q$ is the shear, $\rho_i = v_{ti}/\Omega_i$, $y=r\zeta/q$ is the normalized toroidal angle, $k=nq/r$ is the poloidal-wave number, r is the minor radius, R is the major radius, and m and n are the poloidal and toroidal mode numbers, respectively. The notation $\overline{\cdots} = \oint(dl/v_\parallel)\cdots$ denotes a bounce average, and $[\cdots]$ signifies that the field inside the brackets is to be evaluated at the rational surface Bloch shifted away from the origin by $2\pi m'$.

We take note of the fact that the physical process we envisage entails a wave-particle resonance ($\omega \simeq \bar{\omega}_{di}$) at which rate ions trapped in the wave are ballistically propagated. The evolution of the two-point phase-space density correlation function can therefore be characterized as proceeding along two disparate time scales: a slow, "average" time scale, determined by the ballistic propagation time of particles in the wave, and a fast "relative" time scale, at which rate these latter decorrelate from each other. The physical content of the operators on the left-hand side of Eq. (1) is precisely this decorrelation of two given phase-space elements in the turbulent bath. This "detuning" process occurs because of *relative* precessional (banana-center) magnetic drifts of the two particles, and *relative* turbulent $\mathbf{E}\times\mathbf{B}$ diffusion, as mathematically represented by the triplet nonlinearity. This physical picture is indeed what is borne out when we make a coordinate transformation to wave frame of reference $(y_\pm, E_\pm, r_\pm) = (y_1, E_1, r_1) \pm (y_2, E_2, r_2)$, and use standard iterative techniques of strong-turbulence theory to renormalize the triplet nonlinearity.[6] When this procedure is carried out, Eq. (1) reduces to the following simple form:

$$\left[\frac{\partial}{\partial t} + \bar{v}_{di}\bar{E}_-\frac{\partial}{\partial y_-} - \mathcal{D}_-\frac{\partial^2}{\partial y_-^2}\right]\langle \delta h(1)\delta h(2)\rangle = \mathcal{S}. \quad (4)$$

Here, $\mathcal{D}_- \simeq 2\mathcal{D}[1-\cos(k_0 y_- + r_-/\Delta_0)]$ is the relative spatial diffusion operator,

$$\mathcal{D} = (\pi c/B)^2 \textstyle\sum_m (2\pi m)^2 \sum_{k',\omega'} k'^2 \hat{s}_1^2 g_{k',\omega'}(1)\langle [\overline{\delta\phi(1)}]^2\rangle_{k',\omega'} \quad (5)$$

is the familiar (renormalized) quasilinear diffusion operator which emanates from a one-point renormalized theory, $g_{k',\omega'} = (-i\omega' + i\bar{\omega}'_{di}\bar{E} + \tau_c^{-1})^{-1}$ is the propagator, $\tau_c = (k_0^2\mathcal{D})^{-1}$ is the correlation time of the background turbulence, and $\Delta_0 \simeq 1/k_0\hat{s}$ and $k_0^{-1} \simeq r_+/q\langle n^2\rangle^{1/2}$ characterize the mean-square spread of the phase-space eddy in the radial and poloidal directions, respectively. The physical content of the decorrelation processes can be further distilled by calculation of the time rate of change of the stochastic divergence of particle orbits. If we focus on the second-moment Langevin equations of stochastic motion,[6] it is possible to derive an expression for the time rate of decorrelation:

$$\tau_{\rm tr} = -\tau_c \ln\{2[k_0\bar{v}_{di}\tau_c\bar{E}_- + \tfrac{1}{2}(k_0 y_- + r_-/\Delta_0)]^2 + \tfrac{1}{2}(k_0 y_- + r_0/\Delta_0)^2\}. \quad (6)$$

Through its dependence on τ_c, $\tau_{\rm tr}$ is inversely proportional to the mean square fluctuation amplitude. This has the reasonable interpretation that the larger the fluctuation amplitudes in the spectrum, the more rapidly the particles "detune" from the waves.

Next, we shift focus to the process by which the fine-scale granularity is regenerated. The formulation of the problem derives inspiration from the physics of discreteness as described by the Lenard-Balescu equation. Thus, we decompose the distribution function into a coherent and an incoherent response, $\delta h_{k,\omega} = \delta h^{(c)}_{k,\omega} + \delta\tilde{h}_{k,\omega}$. When substituted into the expression for the source term, Eq. (3), the coherent response will give rise to a term which physically accounts for the relaxation of the average distribution function by diffusive mixing due to the turbulent spectrum of ion fluctuations. The substitution of the incoherent response accounts for the frictional drag exerted on the test trapped-ion "eddy" by the bath of turbulent fluctuations. As in the Lenard-Balescu case, momentum conservation requires that same-species diffusion and drag balance. An expansion free-energy source is made accessible by the left-over piece in the polarization drag, which physically represents the Čerenkov emission induced by the dressed-ion granulation as it moves through the plasma.[7] This free-energy source provides the driving energy for the incoherent fluctuations. After some tedious algebra, we obtain the following expression for the source:

$$\mathcal{S} \simeq n_0 \sum_{k',\omega'} \{\omega' - \omega'_{*i}[1+\eta_i(\bar{E}_+ - \tfrac{3}{2})]\}(\mathrm{Im}\epsilon^{e}_{k',\omega'}/|\epsilon_{k',\omega'}|^2)\langle(\delta\tilde{n}/n_0)(\delta\tilde{h}^*/n_0)\rangle_{k',\omega'}\langle f_i(\bar{E}_+)\rangle, \tag{7}$$

where $\delta\tilde{n} = \int d^3v\, \delta\tilde{h}$ is the incoherent part of the density fluctuations, $\epsilon_{k,\omega} = \epsilon^{i}_{k,\omega} + \epsilon^{e}_{k,\omega}$ is the dielectric function as obtained from quasineutrality, i.e., $\epsilon_{k,\omega} e\delta\phi/T_i = \delta\tilde{n}/n_0$ (in the absence of incoherent fluctuations, one recovers the usual linear dispersion relation, $\epsilon_{k,\omega}=0$), the superscripts e and i refer to electrons and ions, respectively, and $\mathrm{Im}\epsilon^{e}_{k',\omega'} < 0$. Two points are particularly noteworthy. First is the fact that the source term is proportional to electron dissipation. The basic physical reason for this proportionality can be understood as follows: As the ion distribution function relaxes by the scattering of trapped-ion eddies down the average gradient, electrons are impelled to respond in order to redress the charge imbalance, and thereby maintain quasineutrality. Thus, quasineutrality (and by extension, ambipolarity) constrains ion relaxation to scale with electron dissipation. Secondly, the fact that $|\omega|/\omega^{T}_{*i} \sim \bar{\omega}_{di}/\omega_{*i} \sim \epsilon_0 < 1$ ($\epsilon_0 = r/R < 1$ is the inverse aspect ratio) implies that the source term is proportional to the free energy stored in the radial gradient of the trapped-ion average distribution function. Thus, a (positive) source will be available to extract energy from the equilibrium gradient through the ion channel when $\eta_i/\eta_{i,\mathrm{cr}}(\bar{E}_+) > 1$, where $\eta_{i,\mathrm{cr}}(\bar{E}_+) \equiv (\frac{3}{2} - \bar{E}_+)^{-1}$. Note that for any value of η_i, there will be some energy range within which trapped-ion eddies will contribute to the source term. This again is in marked contrast to the predictions of the linear theory, and suggests the possibility of a subcritical (i.e., nonlinear) instability. Our expression for the source term may thus be written in the suggestive form

$$\mathcal{S} \simeq 2\pi n_0 \sum_{k'} \omega'_{*i}[\eta_i/\eta_{i,\mathrm{cr}}(\bar{E}_+) - 1](\mathrm{Im}\epsilon^{e}_{k',\bar{\omega}'_{di}\bar{E}_+}/|\epsilon_{k',\bar{\omega}'_{di}\bar{E}_+}|^2)\langle(\delta\tilde{n}/n_0)(\delta\tilde{h}^*/n_0)\rangle_{k'}\langle f_i(\bar{E}_+)\rangle. \tag{8}$$

It is the incoherent piece of the two-point correlation function which contains information pertinent to singular behavior at short separation. An equation describing its nonlinear dynamics can then be written down:

$$\langle\delta\tilde{h}(1)\delta\tilde{h}(2)\rangle \simeq [\tau_{\mathrm{tr}}/(\gamma_{\mathrm{nl}}\tau_{\mathrm{tr}}+1) - \tau_c/(\gamma_{\mathrm{nl}}\tau_c+1)]\mathcal{S}, \tag{9}$$

where γ_{nl} is the nonlinear growth rate. An approximate, closed-form expression for the nonlinear dispersion relation is obtained upon our relating $\langle\delta\tilde{h}\delta\tilde{h}\rangle$ to $\langle\delta\tilde{n}\delta\tilde{h}\rangle_k$. The result of this exercise is

$$\gamma_{\mathrm{nl}} \simeq \tau_c^{-1}\left[\left|2\frac{\omega}{\bar{\omega}_{di}}\right|^{1/2} \mathcal{N}(k)\left(\frac{\eta_i}{\eta_{i,\mathrm{cr}}(|\omega/\bar{\omega}_{di}|)} - 1\right)\frac{\epsilon^{e}_{\mathrm{Im}}}{|\partial\epsilon_{\mathrm{Re}}/\partial k|\,|\epsilon_{\mathrm{Im}}|}\exp\left(-\left|\frac{\omega}{\bar{\omega}_{di}}\right|\right) - 1\right], \tag{10}$$

where

$$\mathcal{N}(k) = \omega_{*i}\int dy_-\exp(-iky_-)\int d\bar{E}_-(\tau_{\mathrm{tr}} - \tau_c).$$

Note that unlike linear theory, the growth rate here is amplitude dependent. The threshold condition (i.e., $\gamma_{\mathrm{nl}}=0$), however, is *independent* of amplitudes and depends only on global profiles (i.e., through η_i). This suggests that the only mechanism by which this nonlinear instability may be quenched is by profile modification through transport processes.

Estimates of the transport fluxes that may be expected to ensue can be derived by our taking appropriate velocity moments of the nonlinear evolution equation for the equilibrium distribution function. The latter may be derived quite simply by our noting that conservation of the phase-space distribution function along particle orbits constrains the evolution of the equilibrium distribution function to balance exactly that of the fluctuating component, i.e., $\langle f_i\rangle\partial\langle f_i\rangle/\partial t = -\frac{1}{2}\partial\langle\delta f_i^2\rangle/\partial t$. The particle ($\Gamma_i$) and thermal ($Q_i$) fluxes are thus given by

$$\begin{bmatrix}\Gamma_i\\ Q_i\end{bmatrix} \simeq (2\epsilon_0)^{3/2}\pi^{-1/2}T_i^{3/2}L_n n_0 \times\sum_{k'}\frac{\omega'^2_{*i}}{\nu_e}\int d\bar{E}_+\bar{E}_+^{1/2}\begin{pmatrix}1\\ T_i\bar{E}_+\end{pmatrix}\frac{[(\epsilon_0/\tau)\bar{E}_+ + 1 + \frac{3}{2}\eta_e][\eta_i/\eta_{i,\mathrm{cr}}(\bar{E}_+) - 1]}{|\epsilon_{k',\bar{\omega}'_{di}\bar{E}_+}|^2}\left\langle\frac{\delta\tilde{n}}{n_0}\frac{\delta\tilde{h}^*}{n_0}\right\rangle_{k'}, \tag{11}$$

where we have made substitutions for $\epsilon^{e}_{\mathrm{Im}}$. The integrals appearing above are analytically intractable. However, to motivate the scenario of proximity to marginal nonlinear stability, we make an order-of-magnitude estimate by invoking a simple mixing-length argument for the two-point correlation function. We can then estimate an upper-bound magnitude for the flux as

$$\begin{Bmatrix}\Gamma_i\\Q_i\end{Bmatrix}\sim-\begin{Bmatrix}1\\|\omega/\bar{\omega}_{di}|\,T_i\end{Bmatrix}\epsilon_0^{3/2}\frac{\omega_{*i}^{T2}}{\nu_e(k\hat{s})^2}\left(\frac{\eta_i}{\eta_{i,\mathrm{cr}}(|\omega/\bar{\omega}_{di}|)}-1\right)\frac{dn_i}{dr}, \tag{12}$$

where $\omega_{*i}^T=-k\rho_i v_{ti}/2L_T$. Two points emerge from the above expressions. First, we conclude that the thermal flux per unit temperature is larger than the particle flux if $|\omega/\bar{\omega}_{di}|>1$. When juxtaposed with the condition for nonlinear instability, the physically relevant frequency regime becomes $\frac{3}{2}>|\omega/\bar{\omega}_{di}|>1$. Above $\frac{3}{2}$, there is no nonlinear instability; below 1, there is no possibility of nonlinear saturation. The second point to note is that the magnitudes of the fluxes are very large. Indeed, because of the ballooning structure of the modes, these flux estimates significantly exceed corresponding ones for dissipative trapped-electron modes.[8] Taken together, these two points suggest that with the onset of this nonlinear instability, transport processes will set in and act to rapidly reconfigure the equilibrium temperature and density profiles in such a way as to return η_i to marginality. Very large heating would be necessary to push η_i above $\eta_{i,\mathrm{cr}}$, so that we may expect η_i to remain very close to its threshold value. The experimental signature of this resonant instability would be the appearance of electrostatic turbulence centered on $\bar{\omega}_{di}$, coinciding with the rapid ejection of a portion of the trapped-ion population, until such time as η_i returns to marginality.

In summary, by following the self-consistent nonlinear evolution of the two-point correlation function for the trapped-ion distribution function in phase space, we have derived conditions under which a nonlinear (i.e., amplitude-dependent) instability, with a threshold on η_i, may occur. Any departure from this nonlinearly marginal state can be expected to lead to large levels of anomalous heat and particle transport, which by modifying the equilibrium temperature and density profiles, respectively, act as a negative feedback mechanism, and return η_i to its threshold value. In some sense, then, the nonlinear instability is quite robust and accentuates the need for pellet injection in the proposed compact ignition tokamak experiment[9] and other reactorlike environments.

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