Universality of Random-Matrix Predictions for the Statistics of Energy Levels

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(Received 3 November 1987)

Wigner statistics for correlations of matrix eigenvalues are shown to be a property of any matrix ensemble with a density of levels and probability distributions for matrix elements that are smooth. This justifies the universality of level correlations in generic quantum systems, while suggesting that level widths and other eigenvector-dependent statistics are system dependent.

PACS numbers: 03.65.−w, 05.45.+b, 21.10.− k

Wigner argued that the statistics of energy-level positions of complicated systems, within an interval of roughly constant expected density of states, $\rho(E)$, is determined solely by level repulsion.[1] He introduced the notion of ensembles of large matrices (Hamiltonians), whose independent elements were Gaussian distributed, and studied the statistics of their eigenvalue distributions. These Gaussian models were solved completely,[2] and the same statistics were later derived for more general ensembles, including any smooth change in measure on the energy continuum[3] [and, consequently, for any smooth $\rho(E)$] and any fixed mean or variance for each Gaussian random variable.[4] Nuclear-level data have continued to support this picture,[5] and further confirmation has come from atoms,[6] molecules,[7] electrons in conductors,[8] and theoretical studies of nonintegrable systems.[9,10]

In this Letter, we make two observations to amplify Wigner's arguments. First, we show that all Hermitean matrix ensembles whose probability weights are smooth (in a sense we make precise in the text that follows) have the same level statistics over intervals in which $\rho(E)$ is approximately constant. The probabilities need not be Gaussian nor independent nor symmetric under some overall orthogonal or unitary transformation on the matrices. Second, if a single fixed matrix is added to each member of a smooth ensemble, the new ensemble typically is smooth. Taken together, these two points suggest that Wigner's statistics are generic for quantum systems with many levels present in intervals of approximately constant mean density of states. In particular, the level-distribution statistics within such intervals are independent of the dynamics and consequent level correlations on much longer scales.

We first sketch some classic results: Level repulsion is manifest in the Jacobian, $J_\beta[E_i]$, for going from independent matrix elements H_{ij} to eigenvalues E_i and rotations α_{ij}:

$$J_\beta[E_i]=\prod_{i<j}|E_i-E_j|^\beta. \tag{1}$$

If only E's are observed, one integrates over α's and expresses the probability measure as

$$\left(\prod_{i\le j}dH_{ij}\right)\mathcal{P}[H_{ij}]\rightarrow\left(\prod_i dE_i\right)J_\beta[E_i]e^{S[E_i]}. \tag{2}$$

For Wigner's Gaussian ensembles,

$$\mathcal{P}[H]=\exp\{-\mathrm{tr}(HH^\dagger)\}$$

and $S[E]=-\sum E_i^2$. β reflects the dimension of the space of α's; $\beta=1$, 2, or 4 if the Hermitean H's are, in fact, relatively real, complex, or symplectic. The product of energy differences can be written as an exponential and interpreted as a pairwise-repulsive logarithmic potential between energy levels in a one-dimensional statistical mechanics[11]; then $S[E]$ is a potential energy that depends smoothly on the level positions. All statistical measures of level distributions can be expressed in terms of n-level correlations, $\mathcal{P}_n(x_1,\ldots,x_n)$, the probability densities describing the presence of levels at n particular energy values. Only the part of $\exp S[E]$ that is symmetric under permutations of the E_i affects the $\mathcal{P}_n$. For the Gaussian unitary ensemble, $\beta=2$ (as well as Dyson's "circular" unitary ensemble, defined on a finite energy interval with periodic boundary conditions),[12]

$$\mathcal{P}_n(x_1,\ldots,x_n)=\det K(x_a,x_b), \tag{3}$$

where $K(x_a,x_b)$ is the $n\times n$ matrix whose entries are a function $K(x,y)$ evaluated at pairs of the points $x_1,\ldots,x_n$. Thus, for example, $\mathcal{P}_1(x)=\rho(x)=K(x,x)$ and $\mathcal{P}_2(x,y)=\rho(x)\rho(y)-[K(x,y)]^2$ {or for the connected $\mathcal{P}_2^c(x,y)=-[K(x,y)]^2$}. In the limit of large matrices with $\beta=2$, $K(x,y)$ approaches

$$\sin[\rho((x+y)/2)\pi(x-y)]/\pi(x-y).$$

The $\mathcal{P}_n$ of the $\beta=1$ and 4 ensembles can be expressed in terms of products of oscillatory functions that decrease with separation like $|x-y|^{-1}$, but the detailed forms are more complicated.

The above results for the $\beta=2$ Gaussian ensemble were generalized[3] to allow $S[E]$ in Eq. (1) to take the form

$$S[E_i]=\textstyle\sum_i\mu(E_i), \tag{4}$$

where $\mu(E)$ is any function of a single variable that is smooth on the scale of the expected mean level spacing, $\rho(E)^{-1}$. In particular, the consequent $\mathcal{P}_n$ are given by Eq. (3), and $K(x,y)$ assumes the same limiting form as above if (1) the $\rho(x)$ that follows from $\mu(x)$ satisfies $\rho^{-2}d\rho/dx \ll 1$, ensuring there exist intervals with an approximately constant ρ that contain many levels; (2) all the points x_i are within an interval small compared to the scale of variation of $\mu(x)$ and $\rho(x)$; and (3) the energies x_i are expressed in units of $\rho(x)^{-1}$ appropriate to that interval.

If there are of order N levels within the scale of variation of ρ or μ, then the deviations from universality are $O(1/N)$. Analogous results are presumed to hold for $\beta=1$ and 4. This is particularly plausible in the language of the one-dimensional gas for which β is the strength of the pairwise logarithmic repulsion. We proceed with $\beta=2$ because the explicit forms are far simpler and we need the above result as a starting point. And we likewise conjecture that the result we establish holds for a continuous range of β including 1 and 4.

The proof[3] of the above assertions for $\beta=2$ ensembles follows closely one of the methods of exact solution[2] of the $\beta=2$ Gaussian ensemble. One notes that for $\beta=2$ the Jacobian in Eq. (1) is the square of the Vandemonde determinant, $\det(E_i^{j-1})$, $i,j=1,2,\ldots,N$. A factor of $\exp[-\mu(E_i)/2]$ can be absorbed into each row, and then rows can be combined to yield a set of orthonormal functions, $\phi_j(E)$. The $\mathcal{P}_n$ are then given precisely by Eq. (3) with

$$K(E,E')=\sum_{j=1}^{N}\phi_j(E)\phi_j(E')\propto\frac{\phi_{N+1}(E)\phi_N(E')-\phi_N(E)\phi_{N+1}(E')}{E-E'}. \tag{5}$$

The universality follows from the fact that, for large j over limited intervals, all sets of orthogonal functions ϕ_j are approximately sinusoidal.

The freedom to choose μ in Eq. (4) allows one to construct ensembles with any *a priori* desired smooth density ρ, and all such ensembles have the same level statistics within dense intervals of roughly constant ρ. However, choosing μ to fit a given ρ also uniquely determines the very long-scale level statistics. We now prove that Wigner's statistics hold under even less restrictive conditions. In particular, any smooth function $f(E_1, E_2, \ldots, E_N)$ can be used for $S[E]$ in Eq. (2), and, under conditions (1), (2), and (3) above (if we replace μ by f), the $\mathcal{P}_n$'s and K assume their universal forms.

The following outline may elucidate the subsequent sketch of our proof. Since we wish to consider a general, smooth $S[E]$, no closed form of the $\mathcal{P}_n$, analogous to Eq. (3), can be given to be then examined as $N\rightarrow\infty$. Rather, we consider a general $S[E]$ as the sum of the simpler $\sum_i\mu(E_i)$ plus a general perturbation. The $\mathcal{P}_n$ are expressed as a power series in the strength of the perturbation, and we characterize the structure of each term in the power series in terms of the known correlations for $\sum\mu$ alone. We find that the power series only makes sense if the perturbation does not change the density of levels as determined by $\sum\mu$ alone. However, this restriction represents no loss of generality, because the original μ and ρ were themselves arbitrary. Given that the perturbation does not change ρ, the task is to show that all $\mathcal{P}_n$ are unchanged [up to corrections of $O(1/N)$] within roughly constant, dense intervals of ρ. To this end, we first estimate the magnitude (in powers of N) of the various terms that arise and then show that the sum of terms in each given order of the perturbation vanishes as $N\rightarrow\infty$. Consider

$$S[E]=\sum_i\mu(E_i)+\lambda\left[\sum_i\mu'(E_i)+f(E_1,\ldots,E_N)\right], \tag{6}$$

where μ' is chosen so that the addition of $\sum\mu'+f$ does not change $\rho(x)$ as determined by $\sum\mu$ alone. It is sufficient that

$$\mu'(x)=-\int\left[\prod_{i=1}^{N}\frac{\rho(y_i)}{N}dy_i\right]f(y_1,\ldots,y_N)\sum_{j=1}^{N}\frac{\delta(y_j-x)}{\rho(y_i)}, \tag{7}$$

which follows from the standard mean-field[13] determination of $\rho(x)$, where $N=\int dy\,\rho(y)$. By virtue of the results for Eq. (4), it suffices to show that $S[E]$ in Eq. (6) gives the same $\mathcal{P}_n$'s as $\sum\mu$ alone—which we do to each order in the parameter λ.

Consider $\lambda(\sum\mu'+f)$ as a perturbation on $\sum\mu$. The change in any $\mathcal{P}_n$ to a given order in λ is a sum of integrals of products of the unperturbed $\mathcal{P}_n$, which are given by Eq. (3). For example, if f is a symmetric three-level potential, $f=\sum h(E_i,E_j,E_k)$, summed over i,j,k, then to $O(\lambda)$,

$$\mathcal{P}_2(E_a,E_b)=\int[dE_i]J_\beta[E_i]\exp\left[\sum_i\mu(E_i)\right]\sum_{j_1,j_2\neq j_1}\delta(E_{j_1}-E_a)\delta(E_{j_2}-E_b)$$
$$\times\left[1+\lambda\left[\sum_l\int dz\,\delta(E_l-z)\mu'(z)+\sum_{l_1,l_2,l_3}\int\prod_{i=1}^{3}dz_i\,\delta(E_{l_i}-z_i)h(z_1,z_2,z_3)\right]\right].$$

The generic correction term to $\mathcal{P}_n$ involves its n arguments $x_1, \ldots, x_n$ and a set of integrated points z_k; each x_i and z_k appears twice as arguments in some $K(z,z')$, and each z_k appears once as an argument of μ' or f; and some terms contain δ functions in place of K's, identifying two points. The δ-function terms arise because the sums over l's as appear in the example above are unrestricted, while the $\mathcal{P}_n$'s are defined by summing $j_1 \neq j_2 \neq \cdots \neq j_n$. In a useful diagrammatic notation, one can represent the K's (and δ functions) as lines connecting the points x_i and z_k; two line ends (not necessarily from different lines) terminate at each point; and the arguments of μ' and f attach at the points z_k. Only those terms connected by factors of $K(z,z')$ or f are needed in computing the connected $\mathcal{P}_n^c$. The universal behavior of $K(z,z')$ is limited by the above conditions (1), (2), and (3). Since the integrals run over all z_k, we *assume* that, beyond the region given by (1), (2), and (3), $K(z,z')$ continues to decrease roughly like $|z-z'|^{-1}$ or faster.

Because of Eq. (7), every occurrence of $\mu'(z)$ is canceled by the otherwise identical terms in which $\mu'(z)$ is replaced by f, with one of its arguments (each in turn) set equal to z, while all others are contracted with themselves via $K(y_i,y_i)=\rho(y_i)$. The remaining terms only involve f, and each f must appear in ways *other* than that just described. For example, two of the many such terms which contribute to $\mathcal{P}_2^c(x,x')$ to $O(\lambda^2)$ using the simple three-level potential, i.e., $f=\sum h(E_i,E_j,E_K)$, are

$$K(x,z_1)\delta(z_1-z_4)K(z_4,x')K(x',z_2)K(z_2,x)K(z_3,z_3)K(z_5,z_6)K(z_6,z_5)$$

and

$$K(x,z_1)K(z_1,x)K(x',z_2)K(z_2,z_4)K(z_4,x')K(z_5,z_3)K(z_3,z_5)K(z_6,z_6),$$

each integrated with $h(z_1,z_2,z_3)h(z_4,z_5,z_6)dz_1\cdots dz_6$ and each with its own combinatoric coefficient.

The sum of these remaining terms (i.e., the ones other than those that explicitly cancel the occurrences of μ') vanishes in the desired limit, as we show in two steps:

I. Each individual remaining term is at most $O(N^0)$, which can be seen as follows. The overall magnitude of μ' and f is set by our requiring each separately to alter ρ by a position-dependent factor of $O(1)$; far smaller is uninteresting, while far larger would totally dominate over level repulsion and corresponds to a different problem. $d\mu'(x)/dx$ enters linearly in the mean-field equation for ρ and must be $O(1)$ relative to ρ. The smoothness assumption on μ' and f states that their variation (and that of ρ) is characterized by a long length L in units of ρ^{-1}, which we henceforth adopt. Thus, μ'/L is $O(1)$. A simple estimate of the magnitude of any particular term among the corrections to $\mathcal{P}_n^c$ would have an additional L for each occurrence of μ'. [The consequent L^n that appears in $O(\lambda^n)$ in the absence of the compensating f makes the present method insufficient for establishing the universality under smooth changes in μ.] For example,

$$\int dz\,dz'\,\mu'(z)\mu'(z')K(x,z)K(z,z')K(z',x')K(x',x)$$

is $O(L)$ times greater than

$$\int dz\,\mu'(z)K(x,z)K(z,x')K(x',x)\ [=\int dz_1dz_2dz_3h(z_1,z_2,z_3)K(x,z_1)K(z_1,x')K(x',x)K(z_2,z_2)K(z_3,z_3)$$

in our simple three-level potential example]. To make these and our subsequent estimates, one must require that the z integrals exist. The mathematics is simple if one takes $-\infty<z_i<\infty$ and assumes that f does not grow at infinity and ρ is integrable, or, more physically, one may assume that the z_i are limited to a finite interval which contains a large but finite number of levels. The occurrences of f that cancel μ' therefore count similarly. If any one of the self-contracted [via $K(y,y)$] arguments of f is instead connected via K's to some other point or points, the magnitude of the resulting term is reduced by $1/N$ or $1/L$, where N is the integral of ρ. $N/L=O(1)$ is the only case of interest. Hence, to any order in λ, each term remaining, after the cancellation of μ' by part of f, is at most $O(1)$.

II. To any given order in λ, the sum of the remaining connected correction terms, each term of at most $O(1)$, to each of the $\mathcal{P}_n^c$ is $O(1/L)$ for $|x-y|=O(1)$. This cancellation occurs because a constant μ' or f, no matter how large, must give zero net change in any $\mathcal{P}_n^c$, and the cancellation can be seen explicitly as follows: Each correction term is of the form of a set of m integrated points, $z_1, \ldots, z_m$, connected by K's (or δ functions) into loops each containing at least one of the x_i's and a set of other points, variously connected by K's to each other but not connected by a string of K's to any x_i. The two sets are connected by factors of the smooth f. The sum of all terms with m points in the containing x_i loops has a product form: the sum of all possible m-point loops (with various orders, δ functions, and topologies) integrated with the sum of all possible "other points," as described above. Integration over these other points produces a total weight factor $g(z_1, \ldots, z_m)$, smooth with scale L, which includes all f's and is integrated against a sum of products of K's. Each term in the set of integrals over the z_i is no larger than $O(1)$. For example, the

$m=2$ piece of $\mathcal{P}_2^{\xi}(x,x')$ is

$$\int dz\,dz'\,g(z,z')\{[K(x,z)K(z,x')K(x',z')K(z',x)-4\delta(x-z)K(z,z')K(z',x')K(x',x)$$
$$+2K(x,z)K(z,z')K(z',x')K(x',x)-K(x,z)\delta(z-z')K(z',x')K(x',x)$$
$$+\delta(x-z)K(z,z')\delta(z'-x')K(x',x)+\delta(x-z)\delta(z-z')K(z',x')K(x',x)$$
$$-K(x,z)K(z,x)K(x',z')K(z',x')-\delta(x-z)K(z,z)\delta(x'-z')K(z',z')$$
$$+2\delta(x-z)K(z,z)K(x',z')K(z',x')+(z\leftrightarrow z')]+(x\leftrightarrow x')\}/2.$$

The region of integration for which any z_i is $O(L)$ away from the x_i is further suppressed by $1/L$ because of the falloff of K. In the remaining region where all z_i are near the x_i, $g(z_1,\ldots,z_m)$ is approximately $g(\bar{x},\ldots,\bar{x})$, where $\bar{x}$ is an average position among the x_i. In particular, the replacement of $g(z_1,\ldots,z_m)$ by $g(z_1,\ldots,z_m)-g(\bar{x},\ldots,\bar{x})$ would make each term in the set of integrals over z_i $O(1/L)$. However, the sum of all the terms integrated over the full range of each z_i with $g(\bar{x},\ldots,\bar{x})$ must be identically zero because it is the same as adding a constant to $S[E]$, evaluated to $O(\lambda)$. Hence, the total effect of $\mu'+f$ is $O(1/L)$, thus completing the proof of the asserted independence of level statistics on the exact form of the smooth part of the statistical weight, $S[E]$, in Eq. (2).

We consider a statistical property to be generic if it is unaffected by a wide class of perturbations. If a starting ensemble with a probability measure $[dH^{\rm old}]\exp S[H^{\rm old}]$ and a smooth $S[E]$ is transformed by addition of a single, given $\epsilon H'$ to each $H^{\rm old}$, then the new measure

$$dH^{\rm new}\exp S'[H^{\rm new}]=dH^{\rm new}\exp S[H^{\rm new}-\epsilon H'].$$

If ϵ is sufficiently small, nothing changes, while, if ϵ is sufficiently big, $\exp S'[E^{\rm new}]$ approaches a product of δ functions at the eigenvalues of $\epsilon H'$. Of interest is the intermediate range in which the elements of $\epsilon H'$ are comparable to those of a typical $H^{\rm old}$. In terms of eigenvalues and rotations, the new measure is $d\mathbb{E}_i^{\rm new}\,d\alpha\,J_\beta[\mathbb{E}^{\rm new}]\exp S[\alpha^{-1}E^{\rm new}\alpha-\epsilon H']$ where $E^{\rm new}$ is the diagonal matrix with entries $E_i^{\rm new}$. If $S[H^{\rm old}]$ is a smooth function of the matrix elements in $H^{\rm old}$ then, for each α, $S(\alpha^{-1}\mathbb{E}^{\rm new}\alpha-\epsilon H')$ is a smooth function of the energy eigenvalues $E_i^{\rm new}$. The integrations over α thus yield a new probability distribution $dE_i^{\rm new}J_\beta[E^{\rm new}]\times\exp S'(E^{\rm new})$ with a smooth S'. Therefore, Wigner statistics for correlations of eigenvalues are generic,[4,14] in that they persist even as the ensemble is subjected to repeated random perturbations.

Statistical measures of eigenvector distributions, which have been related, for example, to nuclear-level widths,[15] are not generic in this sense. The salient eigenvector features of the Gaussian ensembles (i.e., each component statistically independent and Gaussian distributed) are typically destroyed by shifting of the ensemble by a fixed $\epsilon H'$, as is evident from the new measure exhibited above. In particular, $\epsilon H'$ identifies preferred directions and destroys the orthogonal or unitary symmetry. For a given physical system or systems, crude statistical measures may resemble the Gaussian-ensemble predictions, but closer inspection will almost certainly reveal discrepancies.[10]

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