Novel Approach to Lattice Field Theory

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(Received 15 March 1988)

A novel action-variational approach based on certain natural criteria is proposed. Its accuracy and advantages are demonstrated for U(1) gauge theory in four dimensions.

PACS numbers: 11.15.Ha, 05.50.+q

Monte Carlo simulations of gauge theories have led to interesting results and considerable insight in recent years. A problem which remains is that one cannot get far enough into the weak-coupling region which is of interest for particle physics. Similarly as in lattice-field theories, critical slowing down in statistical physics is also an obstacle to the solution of some interesting problems.

In order to look for a way to obtain nonperturbative results beyond the simulation region, a reconsideration of action-variational approaches appears appropriate. In lattice-gauge theory one begins with mean-field methods[1] and their refinement by a saddle-point expansion.[2] Next, the cumulant expansion of the free energy is used.[3] Then a method which imposes the Schwinger-Dyson equation on an expansion of the correlation function is introduced.[4] With this method we recently were able to calculate to higher orders,[5] determining geometrical coefficients by computer instead of enumerating contributions graphically. Our quantitative second-order results revealed the necessity of revising the rules of this method and showed that severe limitations on its computational accuracy remain. This is because the Schwinger-Dyson equation in many cases has no true solution and one has to rely on a crude approximation.

In the present Letter a different approach is proposed. It starts from the fact that in the calculation of correlation functions the best results are to be expected if the trial action is adjusted for each of these functions separately. This requires that an expansion of the correlation function rather than, say, one of the generating functional, as in the saddle-point method, is to be used. It also excludes the use of Schwinger-Dyson equations in which sums of different functions would have the same trial action. The crucial question then is where to get a criterion for the adjustment of the trial action. The answer is that a natural criterion is already inherent in the procedure and one does not need to introduce an additional device which could fix things at best in an indirect way. This criterion follows from the obvious requirement that the expansion, to some finite order, must be a good approximation. Accordingly, the trial action has to be adjusted to a parameter point where the sequence of finite-order approximations has an accumulation point.

In the following, this approach is demonstrated for U(1) gauge theory in $d=4$ dimensions with computations up to fourth order. Denoting correlation functions with respect to the action S by $\langle\cdots\rangle$ and to the trial action S_0 by $\langle\cdots\rangle_0$, one has the expansion

$$\langle X\rangle=\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\langle(S-S_0)^{\nu}X\rangle_{\delta}, \tag{1}$$

where $\langle\cdots\rangle_{\delta}$ are connected functions. With use of the Wilson form $S=\beta\sum_{\pm p}U_p$ and the standard form $S_0=\alpha\sum_{\pm l}U_l$, where the signs of $\pm p$ and $\pm l$ indicate different orientations of plaquettes and links, respectively, (1) becomes

$$\langle X\rangle=\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\sum_{\mu=0}^{\nu}\binom{\nu}{\mu}(-1)^{\mu}\beta^{\nu-\mu}\alpha^{\mu}F_{\nu-\mu,\mu} \tag{2}$$

with

$$F_{\rho\mu}=\sum_{\pm p_1,\ldots,\pm p_{\rho}}\sum_{\pm l_1,\ldots,\pm l_{\mu}}\langle U_{p_1}\cdots U_{p_{\rho}}U_{l_1}\cdots U_{l_{\mu}}X\rangle_{\delta}. \tag{3}$$

Evaluation of the integrals in (3) leads to the form

$$F_{\rho\mu}=\sum_j c_{\rho\mu j}V_1^{E_{\rho\mu j1}}\cdots V_m^{E_{\rho\mu jm}}, \tag{4}$$

where j numbers different configurations and V_K is the ratio of modified Bessel functions $I_K(2\alpha)/I_0(2\alpha)$. The integer constants $c_{\rho\mu j},E_{\rho\mu j1},\ldots,E_{\rho\mu jm}$ are determined by the possible geometric configurations. They are obtained with a computer algorithm which finds and counts all types of configurations and transforms to the contributions corresponding to connected functions.

When we consider rectangular Wilson loops, $\langle X\rangle$ in (1) is denoted by W and the approximation of (1) in which the sum only extends up to n by W_n. Figure 1 shows typical results for W_0 to W_4 as functions of $r=\alpha/[2(d-1)\beta]$. It is seen that there is, in fact, an accumulation point not far from $r=1$ at which the Monte Carlo data[6] are nicely approached. The property that W does not depend on r is reflected by the W_n becoming less steep for larger n in the vicinity of this point. The picture (in Fig. 1 for a 1×1 loop at $\beta=0.706\,55$) is qualitatively the same for other loop sizes and for other β values

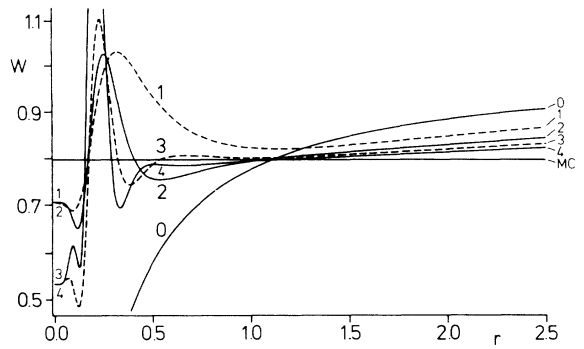

FIG. 1. Typical behavior of W_0 to W_4 as functions of r compared with the Monte Carlo value (shown for a 1×1 loop at $\beta=0.706\,55$).

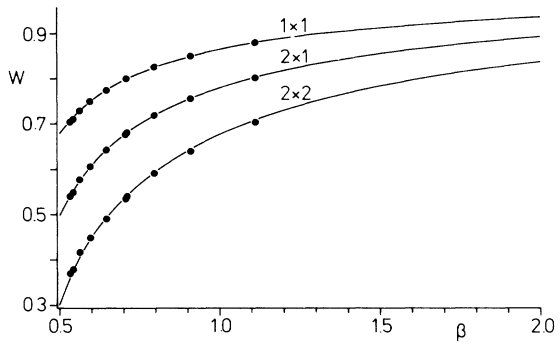

FIG. 2. W_4 for $r=r_a$ and loop sizes 1×1, 2×1, and 2×2 compared with Monte Carlo data.

above 0.5. The r range where this picture occurs, however, changes. In particular, for β increasing from 0.5 to 2, the r value of the accumulation point increases from 0.95 to 1.27 for a loop size 1×1, from 0.81 to 1.11 for 2×1, and from 0.66 to 0.95 for 2×2. This shows that the adjustment of the trial action for each function separately is indeed very important.

Because I have used only a small number of W_n, there remains some freedom in fixing the optimal accumulation point. In order to exploit the available W_n in the best possible way, a prescription is desirable which restricts this freedom. The fact that W, being constant, and W_0, being monotonically increasing, always meet at exactly one point suggests that we require $W=W_0$, which in (1) is seen to correspond to vanishing of the terms which depend on the difference $S-S_0$. If the first up to the nth of these terms sum to zero, in general the higher terms can be expected to give only a small contribution if the expansion makes sense at all. Then $W=W_0$ can be replaced by the approximate condition $W_n=W_0$. An empirical study of the "fine structure" of the accumulation vicinity confirms that this prescription leads systematically to the best results. Therefore it is used to fix the value $r=r_a$ of the accumulation point.

Figure 2 shows the fourth-order results W_4 at $r=r_a$ for β between 0.5 and 2 for loop sizes 1×1, 2×1, and 2×2, and their good agreement with Monte Carlo data.[6] To see the accuracy in detail in Fig. 3 W_1 to W_4 as well as the Monte Carlo values W_{MC} are divided by W_4. It becomes obvious that the accuracy of the Monte Carlo data is reached. It is also seen that the results become more precise for higher β such that the method works particularly well beyond the simulation range. It further turns out that the asymptotic behavior for large β already visible in Figs. 3(a) and 3(b) also occurs for larger loops as soon as β is sufficiently large.

At β values below the transition point, which according to Monte Carlo investigations[7] is at 0.505, the accumulation point considered so far suddenly disappears. Figure 4 shows where this causes the related evaluation

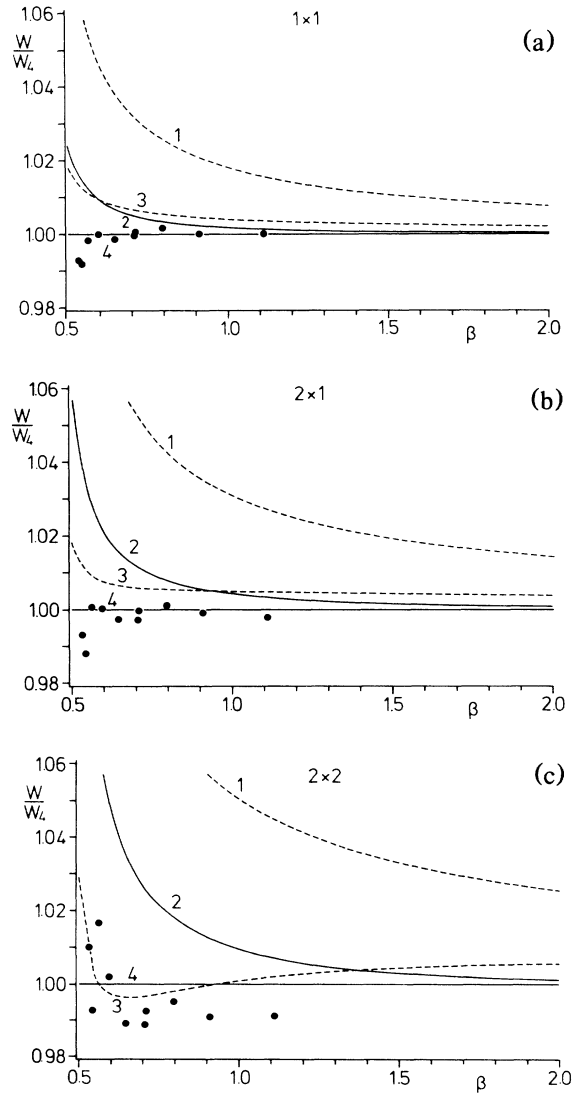

FIG. 3. W_n/W_4 for $n=1$ to 4 and W_{MC}/W_4 with W_1 to W_4 taken at their respective r_a for loop sizes (a) 1×1, (b) 2×1, and (c) 2×2.

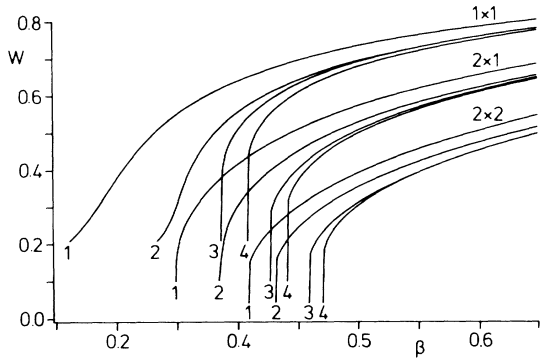

FIG. 4. W_1 to W_4 at their respective r_a for loop sizes 1×1, 2×1, and 2×2 near the transition point.

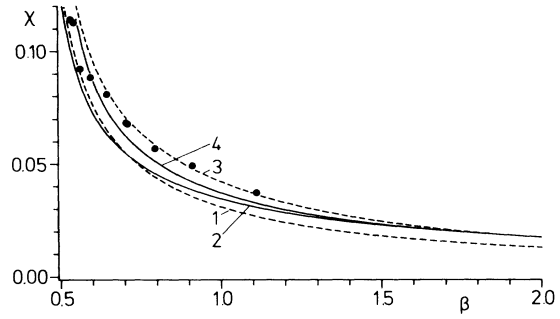

FIG. 5. Creutz ratio χ for $n=1$ to 4 compared with Monte Carlo values.

to break down. It is seen that higher orders and larger loops are more sensitive in this respect. For fourth order and loop size 2×2, one is only 6% below the expected transition point and it is obvious from Fig. 4 that an extrapolation in order or loop size reduces this deviation. Thus it appears that the method provides a clear signal for the transition, and, on the other hand, it is seen to keep its accuracy down to this point.

Below the transition point, though the solution where α is of order $2(d-1)\beta$ is lost, the method still works. Now the accumulation point at $\alpha=0$ (for orders which are able to cover the loop with plaquettes) becomes reasonable. An indication for this that the fluctuations of orders then become smaller than those of the upper solution. On the other hand, this is clear because at $\alpha=0$ one gets just a particular form of strong-coupling expansion. Then in (4) only those terms contribute where $E_{\rho\mu j1}=\cdots=E_{\rho\mu jm}=0$. A consequence of this is that up to fourth order in (2) only 2 of 515, 2 of 740, and 1 of 724 terms remain for loop sizes 1×1, 2×1, and 2×2, respectively. Thus technically it becomes much easier to go to higher orders.

Figure 5 shows the results obtained for the Creutz ratio[8]

$$\chi=-\ln\left[\frac{W(2,2)W(1,1)}{W(2,1)W(1,2)}\right] \tag{5}$$

by inserting the W_n presented in Figs. 2 and 3. It is seen that the deviations from the Monte Carlo values are still moderate and smaller for third and fourth order. For large β the orders $n=2$ to 4 approach each other which indicates better accuracy beyond the simulation region also for Creutz ratios.

Finally, it appears appropriate to stress some main observations. The adjustment of the trial action for each correlation function separately is crucial to obtain quantitative results. The natural criterion based on the fact that the expansion must provide a good approximation is quite stringent, not only allowing one to obtain precise results but also giving one a clear signal for the transition point and the switching between solutions. The accuracy increases for larger β such that one gets a particularly precise tool beyond the simulation region.

I wish to thank Michael Creutz for the Monte Carlo data.

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