Spectrum of Large Random Asymmetric Matrices

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The average eigenvalue distribution $\rho(\lambda)$ of $N\times N$ real random asymmetric matrices J_{ij} ($J_{ji}\neq J_{ij}$) is calculated in the limit of $N\to\infty$. It is found that $\rho(\lambda)$ is uniform in an ellipse, in the complex plane, whose real and imaginary axes are $1+\tau$ and $1-\tau$, respectively. The parameter τ is given by $\tau=N[J_{ij}J_{ji}]_J$ and $N[J_{ij}^2]_J$ is normalized to 1. In the $\tau=1$ limit, Wigner's semicircle law is recovered. The results are extended to complex asymmetric matrices.

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The statistical properties of the eigenvalues of large random matrices have been the focus of great interest in mathematics and physics.[1,2] One of the well-studied ensembles is the Gaussian ensemble of real *symmetric* matrices. In this case, the average density of eigenvalues $\rho(x)$ is given by the celebrated Wigner semicircle law[1-3]:

$$\rho(x)=(2\pi)^{-1}(4-x^2)^{1/2},\quad |x|<2, \tag{1}$$

and $\rho(x)=0$ for $|x|\geq 2$. Here we normalized the second moment of the matrix elements to be $[J_{ij}^2]=N^{-1}$, N being the dimensionality of the matrix.

In this Letter we study the average distribution of eigenvalues of a Gaussian ensemble of large real and complex *asymmetric* matrices. Ensembles of *fully asymmetric* matrices, where J_{ij} and J_{ji} are *independent* random variables, have been studied previously.[1,4] It has been shown that in this case, the average eigenvalue distribution is uniform in a disk, in the complex plane, centered at the origin. This result is generalized in this paper to matrices with arbitrary correlations between J_{ij} and J_{ji}.

The statistical properties of random asymmetric matrices may be important in the understanding of the behavior of certain dynamical systems far from equilibrium. One example is the dynamics of neural networks.[5] A simple dynamic model of neural networks consists of N continuous "scalar" degrees of freedom ("neurons") obeying coupled nonlinear differential equations ("circuit equations"). The coupling between the neurons is given by a synaptic matrix J_{ij} which, in general, is asymmetric and has a substantial degree of disorder. In this case, the eigenstates of the synaptic matrix play an important role in the dynamics particularly when the neuron nonlinearity is not big.[5]

We study an ensemble of $N\times N$ real asymmetric matrices J_{ij} defined by a Gaussian distribution with zero mean and correlations

$$N[J_{ij}^2]_J=1,\quad N[J_{ij}J_{ji}]_J=\tau, \tag{2}$$

for $i\neq j$ and $-1\leq\tau\leq 1$. The brackets $[\cdots]_J$ denote ensemble average. The case $\tau=1$ corresponds to the well-studied ensemble of symmetric matrices.[2] The case $\tau=0$ corresponds to the fully asymmetric ensemble in which J_{ij} and J_{ji} are independent,[4] and $\tau=-1$ corresponds to an ensemble of antisymmetric matrices.

The correlations (2) can be derived from a Gaussian measure,

$$P(\mathbf{J})\propto\exp\left[-\frac{N}{2(1-\tau^2)}\mathrm{Tr}(\mathbf{J}\mathbf{J}^T-\tau\mathbf{J}\mathbf{J})\right], \tag{3}$$

where $J_{ij}^T=J_{ji}$. This measure implies for the diagonal elements $N[J_{ii}^2]_J=1+\tau$. In the large-N limit, the diagonal elements give only an $O(1/N)$ contribution, so that the substitution $J_{ii}=0$ (which is often the case in dynamical systems) will not modify the results in the $N\to\infty$ limit.

Let us denote by $\rho(\omega)$ the average density of eigenvalues at the point $\omega=x+iy$. The main result of this paper is that, in the $N\to\infty$ limit, $\rho(\omega)$ is given by

$$\rho(\omega)=\begin{cases}(\pi ab)^{-1}, & \text{if } (x/a)^2+(y/b)^2\leq 1;\\ 0, & \text{otherwise};\end{cases} \tag{4}$$

where $a=1+\tau$ and $b=1-\tau$. In other words, the average density of eigenvalues of the ensembles of large random asymmetric matrices defined by Eqs. (2) or (3) is uniform in an ellipse, in the complex plane, with semiaxes a (real direction) and b (imaginary direction). In the case of fully asymmetric matrices, i.e., $\tau=0$, the ellipse degenerates into a unit circle.[4]

We note that the projection of $\rho(\omega)$ on the real axis

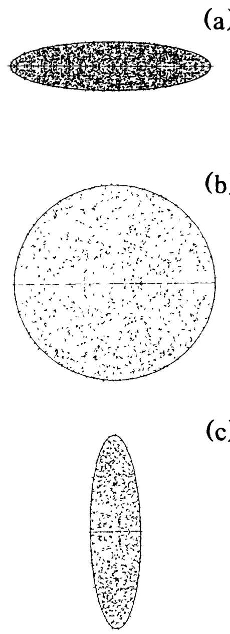

FIG. 1. Numerical results for the distribution of eigenvalues for $N=200$. (a) $\tau=\frac{3}{5}$; (b) $\tau=0$; (c) $\tau=-\frac{3}{5}$. The number of samples is 38, 50, and 25 for (a), (b), and (c), respectively. Each distribution is centered at the origin of the complex plane. The horizontal and vertical directions are the real and imaginary axes, respectively. The full line shows the ellipse predicted by Eq. (4). For graphical reasons we have rescaled the axes of (b) by a factor of $\frac{8}{5}$.

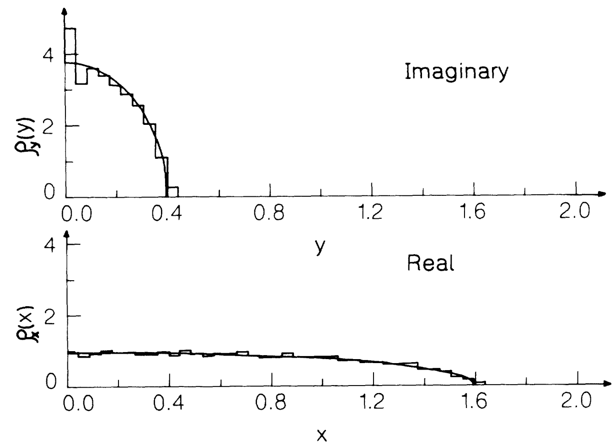

FIG. 2. Histograms of the projections of the distribution on the imaginary and real axes, $\rho_y(y)$ and $\rho_x(x)$, for $N=200$ and $\tau=\frac{3}{5}$. The full line is the generalized semicircle law [see Eq. (5)].

leads to a *generalized semicircle law:*

$$\rho_x(x)\equiv\int dy\,\rho(\omega)=\frac{2}{\pi a^2}(a^2-x^2)^{1/2},\qquad |x|\leq a, \tag{5}$$

where $a=1+\tau$. Obviously a similar law exists along the imaginary axis, i.e., for $\rho_y(y)$. As expected, in the limit of symmetric matrices, $\tau=1$, $\rho(\omega)=\delta(y)\rho_x(x)$ where $\rho_x(x)$ reduces to Wigner's semicircular law, Eq. (1).

In Figs. 1 and 2 we present the results of numerical diagonalization of random asymmetric matrices with $N=200$. As can be seen, the agreement between the numerical results and the analytical predictions (for $N\to\infty$) is very good. The only significant deviation is the nonuniformity of the density of states near the real axis. In fact, the observed density of states on the real axis is higher than the average density, whereas the density slightly above and below the real axis is less than the average. To check that this nonuniformity is a finite-size effect, we have measured the average number of *real* eigenvalues for different sizes N ($N=50$–400). The results, presented in Fig. 3, clearly show that the excess density of real eigenvalues vanishes as $N\to\infty$ (roughly as $N^{-1/2}$). The origin of this excess density is the fact that the *level repulsion* of eigenstates near the real axis is less than the *average level repulsion.*[6] In the following we outline the deviation of Eq. (4).

It is convenient to define the following Green's function:

$$G(\omega)=\frac{1}{N}\left[\mathrm{Tr}\frac{1}{\mathbf{I}\omega-\mathbf{J}}\right]_J, \tag{6}$$

where $\mathbf{I}$ is the identity matrix. This quantity is defined

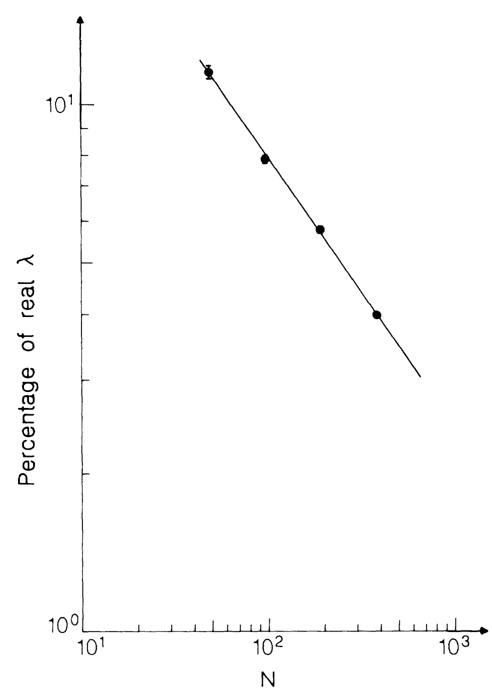

FIG. 3. Numerical results for the average fraction of *real* eigenvalues as a function of N for $\tau=0$. The line shows the slope -0.5.

for all complex values of ω, except for the eigenvalues of J_{ij}. Expanding J_{ij} with the help of a set of right and left eigenvectors, one obtains

$$G(\omega)=\frac{1}{N}\left[\sum_{\lambda}\frac{1}{\omega-\lambda}\right]_J=\int d^2\lambda\frac{\rho(\lambda)}{\omega-\lambda}, \quad (7)$$

where $\rho(\lambda)$ is the average density of eigenvalues λ of J_{ij} in the complex plane. Equation (7) suggests an analogy with two-dimensional classical electrostatics. To show this, we integrate $G(\omega)$ around a region $\mathcal{R}$, assuming that no eigenvalue lies on the border $\partial\mathcal{R}$:

$$\int_{\partial\mathcal{R}}\frac{d\omega}{2\pi i}G(\omega)=\frac{1}{N}\left[\sum_{\lambda}\int_{\partial\mathcal{R}}\frac{d\omega}{2\pi i}\frac{1}{\omega-\lambda}\right]_J$$
$$=\frac{1}{N}\left[\sum_{\lambda\in\mathcal{R}}1\right]_J=\int_{\mathcal{R}}d^2\lambda\rho(\lambda). \quad (8)$$

Noting that $d\omega/i=dy-i\,dx$ is the normal vector to $\partial\mathcal{R}$ and applying Gauss's law, one finds

$$\frac{1}{2\pi}\int_{\mathcal{R}}d^2\omega\left[\frac{\partial}{\partial x}G(\omega)+i\frac{\partial}{\partial y}G(\omega)\right]$$
$$=\int_{\mathcal{R}}d^2\omega\rho(\omega). \quad (9)$$

Since the region $\mathcal{R}$ is arbitrary, it follows that the vector $\mathbf{E}$,

$$E_x\equiv 2\mathrm{Re}G,\quad E_y\equiv -2\mathrm{Im}G, \quad (10)$$

obeys the classical equation of electrostatics, i.e., there exists a potential Φ satisfying

$$2\mathrm{Re}G=-\partial\Phi/\partial x,\quad -2\mathrm{Im}G=-\partial\Phi/\partial y, \quad (11)$$

which obeys Poisson's equation:

$$\nabla^2\Phi=-\nabla\cdot\mathbf{E}=-4\pi\rho. \quad (12)$$

To evaluate $\rho(\omega)$ one has to know $G(\omega)$ in the region where ρ is not zero. In general, it is not possible to evaluate $G(\omega)$ by analytic continuation from outside this region because, in the language of electrostatics, the charge distribution is not completely determined by the value of the electric field outside the charged region. This implies that $G(\omega)$ cannot be calculated by perturbative methods. To demonstrate this point let us expand Eq. (7) in powers of $\mathbf{J}$,

$$G(\omega)=\frac{1}{\omega}\left[1+\frac{1}{N\omega}\sum_i J_{ii}+\frac{1}{N\omega^2}\sum_{i,j}J_{ij}J_{ji}+\cdots\right]_J. \quad (13)$$

If we consider, for example, the fully asymmetric case we have $[J_{ij}J_{ji}]_J=0$, so that the expansion (13) yields $G(\omega)=1/\omega$, in the $N\to\infty$ limit, to all orders in $\mathbf{J}$. However, this result is not valid everywhere. In fact, Eq. (4) implies that for $\tau=0$,

$$G(\omega)=\frac{1}{\pi}\int_{|\lambda|\le 1}\frac{d^2\lambda}{\omega-\lambda}=\begin{cases}\omega^*, & \text{if } |\omega|\le 1;\\ 1/\omega, & \text{if } |\omega|\ge 1.\end{cases} \quad (14)$$

We note that $1/\omega$ corresponds to the two-dimensional Coulomb law, and ω^* corresponds to a linear electric field inside a homogeneously charged disk. Thus Eq. (14) shows that the perturbative result is valid only in the region where $\rho=0$. Note that $G(\omega)$ is not an analytic function of ω inside the disk. It should be stressed that in the case of symmetric (or antisymmetric) matrices the *charge* is concentrated on a line and, therefore, analytic continuation can be used to evaluate $G(\omega)$ and $\rho(\omega)$ everywhere.[2]

The starting point of our approach to determine the spectrum of asymmetric matrices is the calculation of the electrostatic potential Φ defined as

$$\Phi(\omega)=-1/N[\ln\det((\mathbf{I}\omega^*-\mathbf{J}^T)(\mathbf{I}\omega-\mathbf{J}))]_J. \quad (15)$$

With use of the properties $\det(\mathbf{AB})=\det\mathbf{A}\det\mathbf{B}$ and $\det\mathbf{A}^T=\det\mathbf{A}$, it is easy to check that Φ given by Eq. (15) satisfies Eqs. (6) and (11). Since J_{ij} is real, the matrix in the determinant of Eq. (15) is positive semidefinite. Thus, in order to avoid zero eigenvalues, which simply correspond to the case that ω is an eigenvalue of J_{ij}, we add a diagonal matrix $\epsilon\delta_{ij}$ where ϵ is positive and infinitesimal. We may, therefore, represent the determinant by a Gaussian integral over complex variables:

$$\Phi(\omega)=\frac{1}{N}\ln\left[\int\left[\prod_i\frac{d^2z_i}{\pi}\right]\exp\left\{-\epsilon\sum_i|z_i|^2-\sum_{i,j,k}z_i^*(\omega^*\delta_{ik}-J_{ik}^T)(\omega\delta_{kj}-J_{kj})z_j\right\}\right]_J. \quad (16)$$

In writing Eq. (16) we have assumed that the average and the ln operations in Eq. (15) commute, in the $N\to\infty$ limit. We have proved that this is indeed the case using the replica method as will be mentioned at the end.

Carrying out the average over the distribution of J_{ij}, Eq. (3), and neglecting $O(1/N)$ terms, we find[7]

$$\exp(N\Phi)=\int\left[\prod_i\frac{d^2z_i}{\pi}\right]\exp\left\{-N\left[\epsilon r+\ln(1+r)+\frac{rx^2}{1+r(1+\tau)}+\frac{ry^2}{1+r(1-\tau)}\right]\right\}, \quad (17)$$

where $r\equiv(1/N)\sum_i z_iz_i^*$ and $\omega=x+iy$. Equation (17) can be rewritten as an integral over $\sigma\equiv 1/r$,

$$\exp(N\Phi)=\frac{N^N}{\Gamma(N)}\int_0^\infty\frac{d\sigma}{\sigma}\exp\left\{-N\left[\frac{\epsilon}{\sigma}+\ln(\sigma+1)+\frac{x^2}{\sigma+1+\tau}+\frac{y^2}{\sigma+1-\tau}\right]\right\}. \quad (18)$$

Note that for the convergence of the integral we need $\epsilon > 0$; otherwise it diverges logarithmically at $\sigma = 0$.

The integral (18) can be evaluated in the $N \to \infty$ limit by the saddle-point method. The saddle-point equation for σ is

$$\frac{\epsilon}{\sigma^2} = \frac{1}{1+\sigma} - \frac{x^2}{(\sigma+1+\tau)^2} - \frac{y^2}{(\sigma+1-\tau)^2}. \qquad (19)$$

From Eq. (11), the Green's function is given by

$$G(\omega) = x/(\sigma+1+\tau) - iy/(\sigma+1-\tau), \qquad (20)$$

in the $\epsilon \to 0^+$ limit. It is readily seen from Eq. (19) that there is a unique saddle point between the two limits $\sigma = 0$ and $\sigma = \infty$. We are interested only in the value of this saddle point for $\epsilon \to 0^+$. The behavior in this limit depends on the value of x and y. Expansion of ϵ in powers of σ yields

$$\epsilon = \sigma^2 \left[1 - \frac{x^2}{(1+\tau)^2} - \frac{y^2}{1-\tau^2} \right] + O(\sigma^3). \qquad (21)$$

Thus, inside the ellipse whose semiaxes are $a = 1+\tau$ and $b = 1-\tau$, the saddle point is at $\sigma \sim \sqrt{\epsilon}$ and G is given by Eq. (20) with $\sigma = 0$. On the other hand, for (x,y) outside the ellipse, σ remains finite as $\epsilon \to 0^+$, and is given by the solution of Eq. (19) for $\epsilon = 0$. From Eqs. (19) and (20) one can check that outside the ellipse the partial derivatives of G satisfy the Cauchy-Riemann equations. Therefore, to evaluate σ and G outside the ellipse we solve Eq. (19) and (20) (with $\epsilon = 0$) for the special case of $y = 0$. Then, by use of the analyticity of G in that regime, the full result is recovered by analytical continuation, i.e., by replacement of x by ω. This yields

$$G(\omega) = \begin{cases} (\omega/2\tau)[1-(1-4\tau/\omega^2)^{1/2}], & \text{outside,} \\ x/(1+\tau) - iy/(1-\tau), & \text{inside.} \end{cases} \qquad (22)$$

Insertion of this result into Eqs. (10) and (12) leads to Eq. (4).

The result (4) can be generalized to an ensemble of Gaussian *complex* asymmetric matrices. In this case, the invariant Gaussian measure is

$$P(\mathbf{J}) \propto \exp\left[-\frac{N}{1-|\tau|^2} \mathrm{Tr}(\mathbf{J}\mathbf{J}^\dagger - 2\mathrm{Re}\,\tau \mathbf{J}\mathbf{J}) \right]. \qquad (23)$$

In other words, the average density of states depends only on the moments $N[|J_{ij}|^2]_J = 1$ and $N[J_{ij}J_{ji}] = |\tau| e^{2i\theta}$. Using the same method as described above we obtain that the average density of states, in the complex ensemble, is uniform inside an ellipse which is centered at zero and has semiaxes $a = 1+|\tau|$ in the direction θ and $b = 1-|\tau|$ in the direction $\theta+\pi/2$, and is zero outside.

Finally, let us comment about the validity of Eq. (16). The average in Eq. (15) can be performed with the replica method,[3] i.e., $[\ln x]_J = \lim_{n\to 0}([x^n]_J - 1)/n$. This requires integration over n complex Gaussian variables $\{z_i^a\}$, $a = 1, \ldots, n$, instead of one variable as in Eq. (16). However, we have checked that at the $(N \to \infty)$ saddle point the coupling between the different "replicas" vanishes and the replica integral collapses into n integrals of the form (16).

Concluding, let us mention that it would be interesting to extend the present study to more complex statistical properties such as the form of the joint probability distribution of the eigenvalues of random asymmetric matrices as well as the distribution of level spacing in the complex plane. It should also be noted that although we have discussed explicitly the Gaussian ensembles [Eqs. (3) and (23)], one can show that the results (for the $N \to \infty$ limit) are valid for much broader classes of ensembles satisfying Eq. (2).

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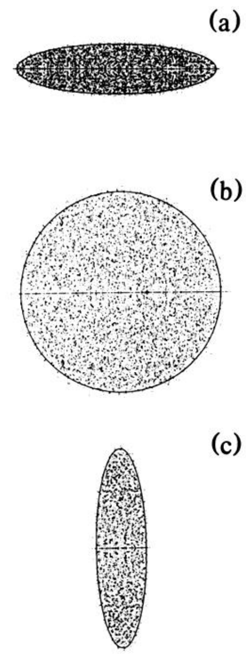
[7]A convenient way of performing the average over J_{ij} in Eq. (16) is by first decoupling the terms quadratic in J_{ij} with a complex Gaussian transformation.



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