Pluralism in the Critical Phenomena of the One-Dimensional Continuous-Spin Ising Model

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A concrete example is given which shows that the one-dimensional Ising- and Gaussian-model universality classes do not exhaust the universality classes of the one-dimensional continuous-spin Ising model. Thus the normal universality hypothesis fails in this simple, readily analyzable model.

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In the modern renormalization-group theory of critical phenomena introduced by Wilson,[1] the concept of universality,[2] i.e., "that apparently dissimilar systems show considerable similarities near their critical points" or more precisely, that critical problems can be divided into classes differentiated only by the dimensionality of the system and the symmetry group of the order parameter (and perhaps other criteria), has been a key concept which has permitted the computation by field-theoretic methods[3] of the universal critical properties of a representative model in each universality class.[4] There has, however, to date been no proof of this universality hypothesis in even the prototype model of the continuous-spin Ising model. The antithesis is the corresponding view of pluralism, i.e., that there exists more than one such universality class. It has been noted[5] that inconsistencies exist in the universality hypothesis, but some[6] have contended that they are not serious (for example, that they may be numerically invisible). The present purpose is to give a concrete example of pluralism.

The model to be considered is the s^4, one-dimensional continuous-spin Ising model. After the work of Isaacson[7] and Marchesin,[8] who showed that the continuum (or critical point) limit of this model was identical to the continuum limit for the pure Ising model, it was generally supposed that there was no problem with universality in this case. This notion was further reinforced by numerical studies[9] of the renormalized coupling constant as a function of the bare coupling constant which showed behavior precisely in line with that anticipated by Ref. 3. The partition function for this model is defined by

$$Z=\int_{-\infty}^{+\infty}\cdots\int \exp\left(K\sum_{i=1}^{N}s_is_{i+1}-\tilde{A}\sum_{i=1}^{N}s_i^2-\tilde{g}_0\sum_{i=1}^{N}s_i^4\right)\prod_{i=1}^{N}ds_i, \tag{1}$$

where K is the exchange integral J over kT, with k the Boltzman's constant and T the absolute temperature. The normalization,

$$\langle 1\rangle=\langle s^2\rangle_{K=0}=\int_{-\infty}^{+\infty}s^2\exp(-\tilde{A}s^2-\tilde{g}_0s^4)ds\Big/\int_{-\infty}^{+\infty}\exp(-\tilde{A}s^2-\tilde{g}_0s^4)ds, \tag{2}$$

is used, which determines $\tilde{A}$ as a function of $\tilde{g}_0$. The special cases ($\tilde{g}_0=0$, $\tilde{A}=\frac{1}{2}$) and ($\tilde{g}_0=\infty$, $\tilde{A}/\tilde{g}_0\rightarrow -2$) are the Gaussian and Ising models, respectively. This model (except for $\tilde{g}_0=0$) has its critical temperature T_c at zero ($K_c=\infty$). For the Ising-model case, its critical behavior is well known. Here as $K\rightarrow\infty$,

$$C_H/N\sim kK^2\operatorname{sech}^2(K),\quad \chi/N\sim m^2e^{2K}/kT,\quad \xi\sim\tfrac{1}{2}e^{2K}, \tag{3}$$

where C_H is the specific heat at constant magnetic field H, χ is the magnetic susceptibility, m is the magnetic moment, and ξ is the true correlation length measured in lattice spacings. We will next see that for the case $0<\tilde{g}_0<\infty$, the model has quite different critical behavior.

To compute the properties of this model, I use the transfer-matrix method. The symmetric transfer matarix here is

$$\begin{aligned}T(t,s)&=\exp[-\tfrac{1}{2}(\tilde{A}t^2+\tilde{g}_0t^4-2Kts+\tilde{A}s^2+\tilde{g}_0s^4)]\\&=\exp\left\{\frac{(K-\tilde{A})^2}{4\tilde{g}_0}-\frac{1}{2}\left[\tilde{g}_0\left(t^2-\frac{K-\tilde{A}}{2\tilde{g}_0}\right)^2+K(t-s)^2+\tilde{g}_0\left(s^2-\frac{K-\tilde{A}}{2\tilde{g}_0}\right)^2\right]\right\}.\end{aligned} \tag{4}$$

In the case where the continuum limit of this model (lattice spacing a goes to zero, with $g_0\propto\tilde{g}_0a^{-3}K^{-2}$ fixed) is taken before the critical-point limit, Isaacson[7] and Marchesin[6] have shown that the first two eigenvalues are asymptotically degenerate and widely separated from the rest of the eigenvalue spectrum. Their arguments are equally valid in the present case in the limit $K\rightarrow\infty$, and so I will need only to consider the largest two eigenstates of $T(t,s)$. To obtain thermodynamic properties we need only[10] $\lim_{N\rightarrow\infty}\ln Z/N=\lambda_1$.

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From the second line of (4) we may write

$$\int_{-\infty}^{+\infty}\int \Psi(t)T(t,s)\Psi(s)dt\,ds$$
$$=\exp\left[\frac{(K-\tilde{A})^2}{4\tilde{g}_0}\right]\int_{-\infty}^{+\infty}\int dt\,ds\left\{\Psi(t)\exp\left[-\tfrac{1}{2}\tilde{g}_0\left(t^2-\frac{K-\tilde{A}}{2\tilde{g}_0}\right)^2-\tfrac{1}{4}K(s-t)^2\right]\right\}$$
$$\times\left\{\Psi(s)\exp\left[-\tfrac{1}{2}\tilde{g}_0\left(s^2-\frac{K-\tilde{A}}{2\tilde{g}_0}\right)^2-\tfrac{1}{4}K(s-t^2)\right]\right\}$$
$$\le\exp\left[\frac{(K-\tilde{A})^2}{4\tilde{g}_0}\right]\int_{-\infty}^{+\infty}\int dt\,ds\,\Psi(t)^2\exp\left[-\tilde{g}_0\left(t^2-\frac{K-\tilde{A}}{2\tilde{g}_0}\right)^2-\tfrac{1}{2}K(s-t)^2\right]$$
$$\le\left(\frac{2\pi}{K}\right)^{1/2}\exp\left[\frac{(K-\tilde{A})^2}{4\tilde{g}_0}\right]\int_{-\infty}^{+\infty}dt\,\Psi(t)^2, \tag{5}$$

where the first inequality follows by the Cauchy-Schwartz inequality and the second from integration over s and noting that when the integrand of one integral is larger at every point than that of another, then the integral is larger as well. It follows directly from (5) that

$$\|T\|\le(2\pi/K)^{1/2}\exp[(K-\tilde{A})^2/4\tilde{g}_0]. \tag{6}$$

By the Rayleigh-Ritz principle any vector $|\Psi\rangle$ gives a lower bound for the largest eigenvalue λ_1 as $\langle\Psi|T|\Psi\rangle/\langle\Psi|\Psi\rangle\le\lambda_1$. Although by the symmetry of T, true eigenvectors are either even or odd in s, it is sufficient for the present to select the trial vector

$$\Psi_+(s)=(2S/\pi)^{1/4}\exp\left(-S\{s-[(K-\tilde{A})/2\tilde{g}_0]^{1/2}\}^2\right), \tag{7}$$

where $S=[(2K-\tilde{A})(K-\tilde{A})]^{1/2}$. Then in the limit as $K\to\infty$, if I evaluate the necessary integrals by the method of steepest descents, I obtain

$$[\pi/(1.5K+S-\tilde{A})]^{1/2}\exp[(K-\tilde{A})^2/4\tilde{g}_0][1+O(K^{-2})]\le\lambda_1\le(2\pi/K)^{1/2}\exp[(K-\tilde{A})^2/4\tilde{g}_0], \tag{8}$$

where the upper bound is from Eq. (6). Thus I conclude that asymptotically as $K\to\infty$

$$\ln\lambda_1=K^2/4\tilde{g}_0-\tilde{A}K/2\tilde{g}_0-\tfrac{1}{2}\ln K+O(1). \tag{9}$$

From (9) we compute directly that

$$C_H\approx kK^2/2\tilde{g}_0, \tag{10}$$

which corresponds to the specific-heat critical index $\alpha=2$ instead of $\alpha=-\infty$ for the Ising model. Here α is the index of divergence of C_H. Later I will use ν to denote the index of divergence of the correlation length.

To compute the critical behavior of the true correlation length, asymptotically as $K\to\infty$, I use

$$\xi\approx-1/\ln(\lambda_2/\lambda_1). \tag{11}$$

The degree of degeneracy between λ_1 and λ_2 can be computed by adaptation of a method of Thompson and Kac.[11] First, however, I note that as remarked above, as $T(t,s)=T(-t,-s)$, the eigenvectors are even or odd in s and so by general principles the eigenvector for λ_1 satisfies $\Psi_1(s)=\Psi_1(-s)$. Let us now choose the trial vector

$$\Phi(s)=\mathrm{sgn}m(s)\Psi_1(s)$$
$$=\Psi_1(s)+[\mathrm{sgn}m(s)-1]\Psi_1(s), \tag{12}$$

which is, by symmetry, orthogonal to $\Psi_1(s)$. Then by the Rayleigh-Ritz principal for the next largest eigenvalue

$$\lambda_2\ge\int_{-\infty}^{+\infty}\int\phi(t)T(t,s)\phi(s)dt\,ds\Big/\int_{-\infty}^{+\infty}\Psi_1(t)^2dt=\lambda_1-4\int_0^{\infty}dt\int_{-\infty}^{0}ds\,\Psi_1(t)T(t,s)\Psi_1(s)\Big/\int_{-\infty}^{+\infty}\Psi_1(t)^2dt, \tag{13}$$

where the second line follows from use of the second form in Eq. (12) and the (s,t) interchange symmetry of $T(t,s)$. By use of the method of Eq. (5) and the inequality[12]

$$e^{x^2}\int_x^{\infty}e^{-t^2}dt\le[x+(x^2+4/\pi)^{1/2}]^{-1}\quad(x\ge0), \tag{14}$$

we can derive from (13)

$$\lambda_2\ge\lambda_1-(2\pi/K)^{1/2},\quad\tilde{A}>0, \tag{15a}$$

$$\lambda_2\ge\lambda_1-(2\pi/K)^{1/2}\exp(\tilde{A}^2/4\tilde{g}_0),\quad\tilde{A}<0. \tag{15b}$$

Thus from (11) and (15), we obtain a lower bound on ξ.

A bound on ξ from above can also be given. Use as a trial vector for λ_1

$$\Phi(s)=\mathrm{sgn}m(s)\Psi_2(s)=\Psi_2(s)+[\mathrm{sgn}m(s)-1]\Psi_2(s), \tag{16}$$

where $\Psi_2(s)$ is the eigenfunction for λ_2. Then by the Rayleigh-Ritz principle again

$$\lambda_1 \geq \int_{-\infty}^{+\infty}\int \Phi(t)T(t,s)\Phi(s)dt\,ds \Big/ \int_{-\infty}^{+\infty}\Psi_2(t)^2dt \geq \lambda_2 - 4\int_0^\infty dt \int_{-\infty}^0 ds\, \Psi_2(t)T(t,s)\psi_2(s) \Big/ \int_{-\infty}^{+\infty}\Psi_2(t)^2 dt$$

$$\geq \lambda_2 + 4\int_b^c\int_b^c dt\,d\sigma\,\Psi_2(t)T(t,-\sigma)\Psi_2(\sigma)\Big/\int_{-\infty}^{+\infty}\Psi_2(t)^2dt. \tag{17}$$

Here I used the fact[7] that $\Psi_2(s)=-\Psi_2(-s)$ and I choose $b=[(K-\tilde{A})/2\tilde{g}_0]^{1/2}-N$ and $c=b+2N$. If N is of order unity then

$$T(t,\tfrac{1}{2}(s_1+s_2))\leq \tfrac{1}{2}T(t,s_1)+T(t,s_2)]$$

for t, s_1, and s_2 in the allowed range of integration. In this range we may, to leading orders in K, approximate Ψ_2 by Ψ_+ of Eq. (7), and replace $T(t,s)$ by the convexity inequality just quoted to give

$$\lambda_1 \geq \lambda_2 + 2(2\pi/S)^{1/2}(1-\epsilon)^2 T([(K-\tilde{A})/2\tilde{g}_0]^{1/2}, -[(K-\tilde{A})/2\tilde{g}_0]^{1/2}), \tag{18}$$

where $\epsilon > 0$ is a small number of order e^{-SN}. Thus

$$\lambda_1-\lambda_2 \geq 2(2\pi/S)^{1/2}(1-\epsilon)^2\exp[-(3K+\tilde{A})(K-\tilde{A})/4\tilde{g}_0]. \tag{19}$$

Hence, by (11), (13), and (19), neglecting corrections to leading orders, we have

$$[K/(3K+2S-2\tilde{A})]^{1/2}\exp\{[(K-\tilde{A})^2/4\tilde{g}_0]-(1-\mathrm{sgn}m\tilde{A})\tilde{A}^2/8\tilde{g}_0\} < \xi < [S/K^{1/2}\exp[K(K-\tilde{A})/\tilde{g}_0]. \tag{20}$$

The significant structure of this one-dimensional model, as far as the limiting field theory is concerned, is identical to that of the Ising model[7,8] and so leads to the same limiting field theory for all $0<\tilde{g}_0\leq\infty$. However, the behavior of the approach to the limiting theory is quite different since $e^{K^2}\gg e^K$ as $K\to\infty$. This difference is consistent with the discovery of Nickel[13] that the critical value g^* of the normalized coupling constant is a singular point of the Callan-Symanzik β function $\beta(g)$, where g^* satisfies $\beta(g^*)=0$. The point $g=g^*$ in the continuum field theory must neccessarily be a point of nonuniform approach in the K-$\tilde{g}_0$ plane for the Callan-Symanzik functions. These functions are related to critical indices which vary with $\tilde{g}_0$, but $g\to g^*$ independent of $\tilde{g}_0>0$ as $K\to\infty$ in this model. Hence the occurrence of a singularity is not surprising in light of the failure of universality in the critical phenomena.

The model further illustrates the point[14] that hyperscaling is not just one question but at least two since the Josephson-Sokal critical index relation $d\nu=2\alpha$ fails while the hyperscaling relations can be shown to hold (properly modified to account for $T_c=0$ and $\langle S_0^2\rangle\to\infty$ as $K\to\infty$) for the higher magnetic derivatives by use of the limiting field theory.

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