Stationary Two-Level Atomic Inversion in a Quantized Cavity Field

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A quantum mechanical analysis of a two-level atom in a coherently driven optical cavity is shown to predict steady-state atomic population inversion. Semiclassically this is forbidden because of the factorization of operator-product expectation values. The full quantum theory is much richer and different field states may be correlated with different atomic states. One consequence is that damping of the cavity field allows atomic polarization to be transferred from higher to lower field states and thus steady-state inversion becomes possible.

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Some of the physics of the electromagnetic field defies classical explanation and can only be understood by our treating the field quantum mechanically. Photon antibunching[1] is an early example. More recently quantum mechanical collapses and revivals in high-Q microwave cavities[2] and squeezing[3] experiments, which may measure the electromagnetic field commutator, have generated great interest. The example I consider here is the interaction of a single two-level atom[4] with a weak quantum mechanical field in an optical cavity. This system has recently been shown to exhibit antibunching, squeezing,[5] and optical bistability.[6] In the "bad cavity" regime I find that when the cavity is illuminated by a laser the system may reach a steady state in which the atom is inverted, that is, the excited-state population exceeds the ground-state population. This is forbidden by the semiclassical theory of the atom-field interaction, which treats the field classically, and may only be understood when both the field and the atom are quantized.

Let us consider two levels of a single atom resonantly coupled to a single Fabry-Perot cavity mode. The excited atom spontaneously emits photons at rate γ and the cavity mode loses photons through the cavity mirrors at rate 2κ. To balance these losses the cavity mode is driven by a resonant laser of amplitude such that in the absence of the atom the cavity mode would equilibrate to a coherent state of real amplitude E/κ. In the weakly driven, bad-cavity regime of this system we expect only a small number of photons in the cavity. Our interest is in such cases having quantum noise sufficiently large that the semiclassical atom-field factorization assumption breaks down.[7]

The dissipation is modeled in a standard way by coupling of the atom and the cavity mode to reservoirs which are subsequently traced over.[8] The resulting interaction-picture master equation for the reduced atom-field density operator ρ is [5-7,9]

$$\begin{aligned} d\rho/dt &= E[a^\dagger - a,\rho] + g[a^\dagger\sigma_- - a\sigma_+,\rho] + L_d\rho, \\ L_d\rho &= \tfrac{1}{2}\gamma(2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-) + \kappa(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a), \end{aligned} \tag{1}$$

where a and $a^\dagger$ are the cavity-mode boson annihilation and creation operators, σ_+ and σ_- are the Pauli atomic raising and lowering operators, and g is the atom-field coupling constant. This master equation is particularly worthy of analysis because it contains all the essential physics of the system; driving, cavity damping, atomic spontaneous emission, and atom-field coupling up to the rotating-wave approximation. I use the conventional parameters: Y the scaled driving-field amplitude, n_s the saturation photon number, and C the single-atom version of the optical bistability cooperativity parameter. They are related to the parameters of Eq. (1) by

$$Y = n_s^{-1/2}E/\kappa, \quad n_s = \gamma^2/8g^2, \quad C = g^2/\kappa\gamma. \tag{2}$$

Equations for the quantum mechanical expectation values of the field amplitude $\langle a\rangle$, for the atomic polarization $\langle\sigma_-\rangle$, which is real because of the various resonance conditions previously stated, and for the inversion $\langle\sigma_z\rangle$ follow from the master equation (1):

$$\begin{aligned} d\langle a\rangle/dt &= E - \kappa\langle a\rangle + g\langle\sigma_-\rangle, \\ d\langle\sigma_-\rangle/dt &= -(\gamma/2)\langle\sigma_-\rangle + 2g\langle a\sigma_z\rangle, \\ d\langle\sigma_z\rangle/dt &= -\gamma(\langle\sigma_z\rangle + \tfrac{1}{2}) - g\langle a^\dagger\sigma_- + \sigma_+ a\rangle. \end{aligned} \tag{3}$$

The usual semiclassical Maxwell-Bloch equations are obtained from these by factorization of the operator-product expectation values, $\langle a\sigma_z\rangle \to \langle a\rangle\langle\sigma_z\rangle$ and $\langle a^\dagger\sigma_-\rangle \to \langle a^\dagger\rangle\langle\sigma_-\rangle$. This factorization is only strictly valid if the field state is independent of the atomic state, $|$ total state $\rangle = |$ atomic state $\rangle|$ field state $\rangle$, or in terms of the density operator, $\rho = \rho_{\text{atom}} \otimes \rho_{\text{field}}$. Alternatively the factorization assumption can be seen as the neglect of quantum noise.[9] With this assumption Eqs. (3) close and may be solved to yield the semiclassical steady-state inversion

$$\langle \sigma_z \rangle = -\frac{1/2}{1+n_s\langle a\rangle^2} \text{ (semiclassical)}, \tag{4}$$

where the mean inversion $\langle \sigma_z \rangle$ is half the difference between the upper- and lower-level populations. Hence with the factorization assumption $\langle \sigma_z \rangle < 0$ and no steady-state inversion is possible. Einstein's A and B coefficient theory of light-matter interactions introduced the processes of absorption, stimulated emission, and spontaneous emission.[10] This theory follows from the factorized form of the Bloch equations (3) on the assumption that $d\langle \sigma_- \rangle/dt = 0$. Then

$$\frac{d\langle \sigma_z \rangle}{dt} = -\gamma(\langle \sigma_z \rangle + \tfrac{1}{2}) - \frac{8g^2}{\gamma}\langle a^\dagger \rangle \langle a \rangle \langle \sigma_z \rangle. \tag{5}$$

The first term on the right-hand side represents spontaneous emission while the term proportional to the field energy density, $\langle a^\dagger \rangle \langle a \rangle$, represents the processes of absorption and stimulated emission. These processes can only increase $\langle \sigma_z \rangle$ if it is negative. If inversion is present $\langle \sigma_z \rangle > 0$ and stimulated emission dominates absorption until $\langle \sigma_z \rangle$ decreases below zero. However, as we shall see, the full quantum theory in a cavity contains additional processes which may dominate those in the Einstein theory, allowing steady-state inversion.

The solution of the master equation (1) is facilitated if we make the canonical transformation to the so-called vacuum picture.[11-13]

$$\begin{aligned} &D^{-1}(E/\kappa)\rho D(E/\kappa) = \rho_v, \\ &D^{-1}(E/\kappa) a D(E/\kappa) = a + E/\kappa, \\ &D(E/\kappa)|0\rangle = |E/\kappa\rangle, \end{aligned} \tag{6}$$

where $D(E/\kappa)$ is the coherent-state displacement operator and $|E/\kappa\rangle$ the coherent state of amplitude E/κ. The master equation (1) then becomes

$$\frac{d\rho_v}{dt} = \frac{gE}{\kappa}[\sigma_- - \sigma_+, \rho_v] + g[a^\dagger \sigma_- - a\sigma_+, \rho_v] + L_d \rho_v. \tag{7}$$

Compared to the original master equation (1) the cavity-mode driving term has been replaced by an atomic driving term. We shall solve the new master equation (7) by taking matrix elements in the basis of Fock field states and atomic eigenstates of σ_z, $\{|n,-\rangle, |n,+\rangle\}$, $n = 0,1,2,\ldots$. In the steady state $d\rho_v/dt = 0$ and we obtain an infinite set of algebraic equations for the density-matrix elements. These may be solved numerically by truncation at some sufficiently high Fock state $|N\rangle$, yielding a set of $(N+1)(2N+3)$ equations for the same number of independent real density-matrix elements. Truncation has previously been used by Carmichael[5] to obtain approximate analytic solutions in the interaction picture. The vacuum-picture master equation is particularly suitable for truncation because in the absence of atom-field coupling, $g = 0$, the vacuum picture steady state is the vacuum state, $\rho_v = |0\rangle\langle 0|$. Of course this corresponds to a coherent state in the interaction picture, $\rho = |E/\kappa\rangle\langle E/\kappa|$. If the atom's presence perturbs the field only slightly we might expect only a few additional Fock states to be necessary for an accurate representation of ρ_v, and this is in fact true in the regime of interest to us.

TABLE I. Steady-state, diagonal, vacuum-picture, density-matrix elements in the basis of field Fock states and atomic eigenstates of σ_z. Parameters are $C = 2$, $n_s = 0.01$, and $Y = 20$. The column sums show an excess of population in the excited atomic state.

n	$\langle n,-\|\rho_v\|n,-\rangle$	$\langle n,+\|\rho_v\|n,+\rangle$	$\langle n,-\|\rho_v\|n,-\rangle + \langle n,+\|\rho_v\|n,+\rangle$
0	0.423	0.485	0.908
1	0.058	0.030	0.088
2	0.003	6×10^{-4}	0.004
3	5×10^{-5}	5×10^{-6}	$< 10^{-4}$
4	4×10^{-7}	2×10^{-8}	$< 10^{-6}$
5	1×10^{-9}	6×10^{-11}	$< 10^{-8}$
Sum	0.484	0.516	1

Our criteria for acceptance of an N for truncation are that the diagonal density-matrix elements be negligible for $n \cong N$ and that the solution does not change when N is increased. The resulting solutions agree with those found by the solution of the dynamical master equation as described in Ref. 6. As a nonnumerical check I have found agreement with the approximate analytic solutions of Rice and Carmichael[7] in the appropriate regime.

Table I shows the diagonal density-matrix elements of a steady-state solution to Eq. (7) obtained with truncation $N = 5$. This system is in the ground field Fock state with about 90% probability and the probability to be in any field state other than the first three, $|0\rangle$, $|1\rangle$, and $|2\rangle$, is less than 0.01%. Hence the truncation at $N = 5$ is valid. The parameters of Table I yield a steady-state atomic inversion of $\langle \sigma_z \rangle = 0.016$, corresponding to about 7% more population in the excited state than in the ground state.

Having demonstrated the existence of steady-state inversion for a particular example we now investigate the physical mechanism responsible. First note that the solution of the vacuum-picture master equation (7) with the two-state truncation, $\{|0,-\rangle, |0,+\rangle\}$, yields the semiclassical inversion, Eq. (4), with $\langle a \rangle = E/\kappa$, the empty-cavity field amplitude. Hence for atom-field couplings sufficiently weak that the cavity field is nearly in the empty-cavity state the two-state truncation approximates the semiclassical inversion. The level of approximation which enables us to understand the origin of steady-state inversion is the four-state truncation, $\{|0,-\rangle, |0,+\rangle, |1,-\rangle, |1,+\rangle\}$. Figure 1 shows the

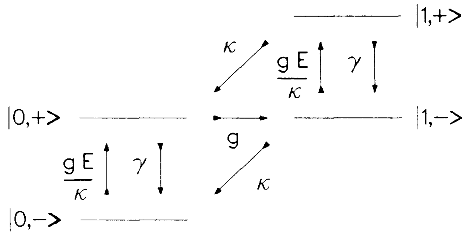

FIG. 1. The four states of the four-state truncation showing directions of steady-state population flow due to the various terms in the master equation (7). Parameters as for Table I.

steady-state population flows due to the various terms in the master equation (7). The coherent flow, proportional to gE/κ, from the less populated state $|0,-\rangle$ to the more populated state $|0,+\rangle$ stands out as contrary to semiclassical expectations. Semiclassically the atomic polarization has the sign required to equalize population differences, which is not the case here. Also the transition would be highly saturated making both the population difference and polarization nearly zero. We can understand the reversed sign of the atomic polarization by looking at the four-state-truncation quantum corrections to the semiclassical picture. From the master equation (7) a steady-state equation for the polarization is

TABLE II. Steady-state inversion $\langle\sigma_z\rangle$ for various values of C and n_s maximized with respect to the scaled driving field Y. The corresponding cavity finesse F and cavity length L are given under the assumption of the sodium D transition and a Gaussian-mode beam waist of three wavelengths.

C	n_s	F	L (mm)	$\langle\sigma_z\rangle$
0.5	0.02	840	0.7	0.0049
0.5	0.1	840	3.5	0.0017
1	0.02	1700	0.7	0.01
1	0.1	1700	3.5	0.0015
2	0.01	3400	0.35	0.016
2	0.05	3400	1.75	0.006
12	0.001	20000	0.035	0.028

$$\tfrac{1}{2}\gamma\langle 0,-|\rho_v|0,+\rangle=(gE/\kappa)[\langle 0,+|\rho_v|0,+\rangle-\langle 0,-|\rho_v|0,-\rangle]-g\langle 0,-|\rho_v|1,-\rangle+2\kappa\langle 1,-|\rho_v|1,+\rangle. \quad (8)$$

Semiclassically we have only the first term, proportional to the population difference. The second term is negligible in the present case. The third term arises from cavity damping and represents the transfer of atomic polarization correlated with the $|1\rangle$ field state into the $|0\rangle$ field state, when the cavity mode loses a photon. When the field is in the $|1\rangle$ Fock state the atomic state is not inverted and the atomic polarization has the sign to move population upwards, as expected. When the cavity mode loses a photon out a mirror this atomic polarization contributes to that in the vacuum-picture ground field state $|0\rangle$. Under highly saturated conditions and with most probability in the vacuum-picture field ground state, this transferred polarization may be sufficient to dominate the small "semiclassical" polarization and thus change its sign. Insofar as the four-state truncation is a good approximation we now have a simple picture of the origin of steady-state inversion. Cavity damping transfers "upward moving" atomic polarization into the inverted ground field state, dominating the small semiclassical polarization. An analogous mechanism of transfer of quantum coherence by a dissipative process, spontaneous emission, is well known in the dressed-atom theory of resonance fluorescence.[14]

Let us now consider whether the necessary conditions for steady-state inversion are experimentally achievable. The inversion only occurs for very small saturation photon numbers, $n_s \ll 1$, corresponding to the strong atom-field coupling associated with small cavity-mode volume. Hence the spherical-mirror Fabry-Perot cavity will require tight focusing and a short length. If we take as our transition the sodium D line, $\lambda=0.6$ μm, $\gamma=6\times10^7$ s^{-1}, the cavity finesse $F=\pi c/2L\kappa$ and length L are related to C and n_s by

$$F=187C(w_0/\lambda)^2,$$
$$L=0.32n_s(\lambda/w_0)^2, \quad (9)$$

where w_0/λ is the beam waist radius in wavelength units. Table II shows the steady-state inversion achievable for various values of C and n_s by maximization with respect to the driving field Y. The corresponding cavity parameters are also given, with the assumption of a beam waist of three wavelengths. For $C=0.5$ and $n_s=0.02$ the cavity parameters are close to reported values.[15] However, the narrow beam waist required implies beam transit times so short that only slow atoms will have time to reach the steady state. Although I have not conducted an exhaustive search for the largest inversion possible, the final value in Table II, $\langle\sigma_z\rangle=0.028$, was among the largest found. This preliminary analysis appears favorable and suggests that more detailed consideration of the experimental feasibility is warranted.

I have presented accurate numerical solutions of the fully quantum mechanical master equation for a single atom in a weakly driven single-mode cavity. In the bad-cavity regime the quantum noise is high, the semiclassical analysis fails, and the steady-state atomic inversion becomes possible. The mechanism can be understood by the consideration of those quantum corrections to the semiclassical behavior which are explicitly due to violation of the semiclassical factorization assumption. Specifically, atomic polarization is transferred to the semiclassically saturated ground field state by cavity damping. A preliminary analysis indicates that steady-state inversion might occur under feasible experimental conditions. Extension of the present work to the case of more than one atom in the cavity and to investigation of possible laser action by the inverted atoms is both feasible and interesting.

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