Accretion onto a Moving Black Hole: An Exact Solution

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We present an analytic solution for the steady-state, subsonic accretion of a gaseous medium onto a Schwarzschild or Kerr black hole. The black hole moves at a constant velocity through the medium, which is uniform and at rest far from the hole and obeys a $P=\rho$ adiabatic equation of state. In the case of a rotating Kerr black hole, the flow is fully three dimensional, but the accretion rate does not depend on the orientation of the hole's spin with respect to the incident direction of the flow.

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Accretion of gas onto astronomical objects is an important phenomenon of long-standing interest to astrophysicists. There are many environments where such accretion provides the underlying source of energy for the emitted radiation. Examples include accretion onto compact objects in binary star systems, accretion onto compact objects moving through the interstellar medium, and accretion onto supermassive black holes in the cores of active galactic nuclei and quasars.[1]

Consider a black hole moving at constant velocity through a gaseous, adiabatic medium at rest and with uniform density at infinity. Determining the steady-state flow poses a classic problem in accretion theory. The Newtonian version of this problem—accretion onto a Newtonian point mass moving nonrelativistically through a nonrelativistic gas—was first discussed by Bondi and Hoyle,[2] but only in qualitative terms. Only in the limit of spherical accretion, appropriate for a stationary black hole, have exact solutions been found (see Bondi[3] for the Newtonian solution or Ref. 1 and Michel[4] for the solution in general relativity).

In general, numerical approaches are required to handle nonspherical accretion for either Newtonian or relativistic flow. However, we have found one exact, fully relativistic, nonspherical solution which may provide valuable physical insight into the more general cases and serve as a benchmark for the testing of numerical codes. Our solution is for a black hole moving through a medium obeying a stiff $P=\rho$ equation of state. The black hole may be either Schwarzschild or Kerr type. As the sound speed is equal to the speed of light, the flow is everywhere subsonic and the solution has no Newtonian analog. Amazingly, our analysis permits the angle between the angular momentum vector of the black hole and direction of the incident flow to be arbitrary. Consequently, the solution can serve as a unique diagnostic not only of spherical and axisymmetric, but also of fully *three-dimensional*, hydrodynamic codes in general relativity.

Just as in Newtonian fluid mechanics, the velocity of a relativistic perfect fluid can be expressed as the gradient of a potential if the vorticity is zero.[5] The relativistic vorticity tensor is defined as

$$\omega_{\mu\nu}=P_\mu^\alpha P_\nu^\beta[(hu_\alpha)_{;\beta}-(hu_\beta)_{;\alpha}], \tag{1}$$

where u^μ is the four-velocity, $h\equiv(\rho+P)/n$ is the enthalpy, and $P_\mu^\nu=\delta_\mu^\nu+u_\mu u^\nu$ is the projection tensor. If the fluid is perfect, Euler's equations become

$$(hu_\mu)_{;\alpha}u^\alpha+h_{,\mu}=0. \tag{2}$$

Equations (1) and (2) yield a simple expression for the vorticity[5]:

$$\omega_{\mu\nu}=[(hu_\mu)_{;\nu}-(hu_\nu)_{;\mu}]. \tag{3}$$

Thus if the vorticity is zero, then the quantity hu_μ can be expressed as the gradient of a potential:

$$hu_\mu=\psi_{,\mu}. \tag{4}$$

As in Newtonian flow, if the vorticity is zero on some initial hypersurface, it will be zero everywhere.

The equation of continuity for the particle (e.g., baryon) density n is $(nu^\alpha)_{;\alpha}=0$ or

$$[(n/h)\psi^{,\alpha}]_{;\alpha}=0. \tag{5}$$

The equation of state relates n to h, and h is found from the normalization equation $h=(-\psi^{,\alpha}\psi_{,\alpha})^{1/2}$, which follows from Eq. (4).

Thus Eq. (5) is, in general, a nonlinear equation in ψ and its derivatives. However, if h is proportional to n, it becomes a linear equation[6]—the equation for a massless scalar field. This simplification occurs if $P=\rho\propto n^2$, which implies that the speed of sound is equal to the speed of light and that the adiabatic index is equal to 2. The flow velocity must everywhere be subsonic and, hence, no shock waves arise. We thus have to solve the equation

$$\psi^{;\alpha}{}_{\alpha}=0 \tag{6}$$

with appropriate boundary conditions.

We analyze the flow in the black-hole rest frame and assume a homogeneous fluid moving at constant velocity at large distances. We seek a stationary solution. Constant velocity and homogeneity upstream imply zero vorticity everywhere, so that the problem reduces to solving

Eq. (6). An important result that we hope to derive is the particle accretion rate,

$$\dot{N} = -\int_S n u^i \sqrt{-g}\, dS_i = -\int_S \psi_{,r} g^{rr} \sqrt{-g}\, d\Omega. \tag{7}$$

Here we have set $n=h$ in appropriate units, and taken the boundary two-surface S to be a sphere centered on the black hole.

Since the medium is homogeneous at large distances, we can set $n=h=1$ there and restore n_∞ later. The asymptotic boundary condition in rectangular coordinates is $\psi = u_\mu x^\mu = -u^0_\infty t + \mathbf{u}_\infty \cdot \mathbf{x}$ or, in spherical coordinates,

$$\psi = -u^0_\infty t + u_\infty r[\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi-\phi_0)] \quad (r\to\infty). \tag{8}$$

We allow the asymptotic three-velocity vector $\mathbf{v}_\infty$ to point along an arbitrary direction (θ_0,θ_0). Note that

$$u^\mu_\infty = (u^0_\infty, \mathbf{u}_\infty) = (1-v^2_\infty)^{-1/2}(1,\mathbf{v}_\infty). \tag{9}$$

Our other boundary conditions are that n and h be finite everywhere, including at the event horizon of a black hole, and that the flow be into, and not out from, a black hole.

Now consider flow onto a Schwarzschild black hole, for which Eq. (6) becomes

$$\psi^{;\alpha}{}_{\alpha} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[\left(1-\frac{2M}{r}\right) r^2 \frac{\partial\psi}{\partial r}\right] + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right] - \left(1-\frac{2M}{r}\right)^{-1}\frac{\partial^2\psi}{\partial t^2} = 0. \tag{10}$$

For stationary flow the gradient of ψ must be independent of time, and so the solution of Eq. (10) can be written as

$$\psi = -u^0_\infty t + \textstyle\sum_{l,m} A_{lm} R_l(r) Y_{lm}(\theta,\phi), \tag{11}$$

where the radial part satisfies

$$\frac{1}{r^2}\frac{d}{dr}\left[\left(1-\frac{2M}{r}\right) r^2 \frac{dR_l}{dr}\right] - \frac{l(l+1)}{r^2} R_l = 0. \tag{12}$$

The general solution of Eq. (12) can be written as a superposition of Legendre functions, $R_l = AP_l(\xi) + BQ_l(\xi)$ where $\xi \equiv r/M - 1$. Thus the general solution for ψ for a Schwarzschild black hole is

$$\psi = -u^0_\infty t + \textstyle\sum_{l,m} [A_{lm} P_l(\xi) + B_{lm} Q_l(\xi)] Y_{lm}(\theta,\phi), \tag{13}$$

where A_{lm} and B_{lm} are constants to be determined from the boundary conditions. The velocities follow from Eq. (4):

$$n u_t = -u^0_\infty, \tag{14a}$$

$$n u_r = \frac{1}{M}\sum_{l,m} [A_{lm} P_l'(\xi) + B_{lm} Q_l'(\xi)] Y_{lm}(\theta,\phi), \tag{14b}$$

$$n u_\theta = \sum_{l,m} [A_{lm} P_l(\xi) + B_{lm} Q_l(\xi)] \frac{\partial Y_{lm}(\theta,\phi)}{\partial\theta}, \tag{14c}$$

$$n u_\phi = \sum_{l,m} [A_{lm} P_l(\xi) + B_{lm} Q_l(\xi)] \frac{\partial Y_{lm}(\theta,\phi)}{\partial\phi}, \tag{14d}$$

with the normalization condition yielding

$$n^2 = (1-2M/r)^{-1}(n u_t)^2 - (1-2M/r)(n u_r)^2 - r^{-2}(n u_\theta)^2 - r^{-2}\csc^2\theta (n u_\phi)^2. \tag{15}$$

Our first important constraint is the finiteness of n at the horizon, $r=2M$. Equation (15) appears divergent at the horizon, but this is a divergence that can be eliminated if one of the spatial velocities is appropriately divergent there. Using the limiting behavior of the Legendre functions near the horizon, $\xi=1$, we find that

$$n^2 \to \left(1-\frac{2M}{r}\right)^{-1}\left[(u^0_\infty)^2 - \left(\frac{1}{4M}\sum_{l,m} B_{lm} Y_{lm}(\theta,\phi)\right)^2\right] \tag{16}$$

near the horizon, which implies that $| B_{00} Y_{00} = 4Mu^0_\infty$, with all the other B's zero. Since we are interested in inward accretion, we select the positive sign for B_{00}. Equation (13) reduces to

$$\psi = -u^0_\infty t - 2Mu^0_\infty \ln(1-2M/r) + \textstyle\sum_{l,m} A_{lm} P_l(\xi) Y_{lm}(\theta,\phi), \tag{17}$$

where the A_{lm} must now be found from the asymptotic boundary conditions in Eq. (8). Without loss of generality, we can specialize to $\theta_0 = 0$, flow toward the north pole of the coordinate system. All the A's vanish except A_{10}, and we have, as our final solution,

$$\psi = -u^0_\infty t - 2Mu^0_\infty \ln(1-2M/r) + u_\infty (r-M)\cos\theta \tag{18}$$

with velocities

$$nu_t = -u_\infty^0, \tag{19a}$$

$$nu_r = -4M^2 u_\infty^0/[r(r-2M)] + u_\infty \cos\theta, \tag{19b}$$

$$nu_\theta = -u_\infty (r-M)\sin\theta, \tag{19c}$$

$$nu_\phi = 0, \tag{19d}$$

and density in units of n_∞

$$n^2 = (u_\infty^0)^2[1+2M/r+(2M/r)^2+(2M/r)^3] - (u_\infty)^2[1-2M/r+(M/r)^2\sin^2\theta] + 8(M/r)^2 u_\infty u_\infty^0 \cos\theta. \tag{20}$$

As can be seen from the above equation, n is clearly finite at the horizon. This solution agrees with the solution given in Ref. 1 for spherical accretion with $\gamma=2$ and $a_\infty=1$ in the limit of $v_\infty=0$. The stagnation point, where the velocity is zero, lies at $\theta=0$ (directly downstream) and at radius

$$r = M[1+(1+4/v_\infty)^{1/2}]. \tag{21}$$

The accretion rate is, from Eq. (7),

$$\dot{N} = 16\pi M^2 n_\infty u_\infty^0. \tag{22}$$

This result is essentially the spherical value multiplied by a Lorentz γ factor for the flow at large distances.

We display a plot of the velocity field for an illustrative case ($v_\infty=0.6$) in Fig. 1.

Now turn to a Kerr black hole. Use the standard form of the metric in Boyer-Lindquist coordinates[7] to write Eq. (6) as

$$-\frac{1}{\Delta}\left[(r^2+a^2)\frac{\partial}{\partial t}+a\frac{\partial}{\partial\phi}\right]^2\psi + \frac{1}{\sin^2\theta}\left(\frac{\partial}{\partial\phi}+a\sin^2\theta\frac{\partial}{\partial t}\right)^2\psi + \frac{\partial}{\partial r}\left(\Delta\frac{\partial\psi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) = 0, \tag{23}$$

where $\Delta = r^2-2Mr+a^2$. The separation of variables in Eq. (11) still works, but with the radial function depending on m as well:

$$(d/dr)(\Delta dR/dr) + [-l(l+1)+m^2a^2/\Delta]R = 0. \tag{24}$$

If we set $r = M+\xi\sqrt{M^2-a^2}$, we see that Eq. (24) is Legendre's equation with an *imaginary* second index. Define $\alpha = a(M^2-a^2)^{-1/2}$. Then the general solution for ψ is

$$\psi = -u_\infty^0 t + \textstyle\sum_l [A_l P_l(\xi) + B_l Q_l(\xi)] Y_{l0}(\theta,\phi) + \sum'_{l,m} [A_{lm}^+ P_l^{im\alpha}(\xi) + A_{lm}^- P_l^{-im\alpha}(\xi)] Y_{lm}(\theta,\phi), \tag{25}$$

where the $'$ in the second sum denotes omission of $m=0$.

The associated Legendre function can be written in terms of a hypergeometric function as[8]

$$P_l^{im\alpha}(\xi) \propto e^{im\chi} F(-l,l+1;1-im\alpha;(1-\xi)/2), \tag{26}$$

where

$$\chi = \tfrac{1}{2}\alpha \ln[(\xi+1)/(\xi-1)] = \tfrac{1}{2}a(M^2-a^2)^{-1/2}\ln[(r-r_-)/(r-r_+)]. \tag{27}$$

Here $r_\pm = M \pm (M^2-a^2)^{1/2}$ are the locations of the event horizons, solutions of $\Delta=0$. Since l is an integer, the hypergeometric function in Eq. (26) is a polynomial in ξ of order l.

The normalization equation for n gives

$$n^2 = (\Sigma\Delta)^{-1}[(r^2+a^2)u_\infty^0 - a(nu_\phi)]^2 - (\Sigma\sin^2\theta)^{-1}[(nu_\phi) - a\sin^2\theta(u_\infty^0)]^2 - (\Delta/\Sigma)(nu_r)^2 - \Sigma^{-1}(nu_\theta)^2, \tag{28}$$

where $\Sigma \equiv r^2+a^2\sin^2\theta$. As for the Schwarzschild black hole, we require n to be finite at the event horizon. We find, in the limit $r \to r_+$,

$$n^2 \to \frac{1}{\Sigma\Delta}\left\{\left[(r_+^2+a^2)u_\infty^0 - a\sum_{l,m}{}' im(A_{lm}^+ e^{im\chi} + A_{lm}^- e^{-im\chi})Y_{lm}(\theta,\phi)\right]^2 - \left[-(M^2-a^2)^{1/2}\sum_l B_l Y_{l0}(\theta,\phi) + a\sum_{l,m}{}'(-A_{lm}^+ e^{im\chi} + A_{lm}^- e^{-im\chi}) im Y_{lm}(\theta,\phi)\right]^2\right\}. \tag{29}$$

Since we want a solution with inflow, we set the sign of B_0 to be positive. We find that all the other B_l's vanish, and also all the A^+_{lm}'s, so that irregular terms can exactly cancel in Eq. (29). Equation (25) for ψ reduces to

$$\psi = -u^0_\infty t + \frac{(r_+^2+a^2)u^0_\infty}{2(M^2-a^2)^{1/2}} \ln\frac{r-r_-}{r-r_+} + \sum_{l,m} A_{lm} F(-l,l+1;1+ima;(1-\xi)/2) Y_{lm}(\theta,\phi-\chi). \tag{30}$$

We now find the remaining coefficients by matching to the asymptotic value given in Eq. (8). No assumption is made concerning the direction of the flow relative to the hole's rotation axis (the polar axis). Only the $l=1$ terms are nonzero, yielding

$$\psi = -u^0_\infty t + \frac{(r_+^2+a^2)u^0_\infty}{2(M^2-a^2)^{1/2}} \ln\frac{r-r_-}{r-r_+} + u_\infty (r-M)\cos\theta\cos\theta_0 + u_\infty \mathrm{Re}[(r-M+ia)\sin\theta\sin\theta_0 e^{i(\phi-\phi_0-\chi)}]. \tag{31}$$

The velocity components are thus

$$nu_t = -u^0_\infty, \tag{32a}$$

$$nu_r = -(r_+^2+a^2)u^0_\infty/\Delta + u_\infty\cos\theta\cos\theta_0 + u_\infty \mathrm{Re}\{[1+ia(r-M+ia)/\Delta]\sin\theta\sin\theta_0 e^{i(\phi-\phi_0-\chi)}\}, \tag{32b}$$

$$nu_\theta = -u_\infty(r-M)\sin\theta\cos\theta_0 + u_\infty \mathrm{Re}[(r-M+ia)\cos\theta\sin\theta_0 e^{i(\phi-\phi_0-\chi)}], \tag{32c}$$

$$nu_\phi = -u_\infty \mathrm{Im}[(r-M+ia)\sin\theta\sin\theta_0 e^{i(\phi-\phi_0-\chi)}]. \tag{32d}$$

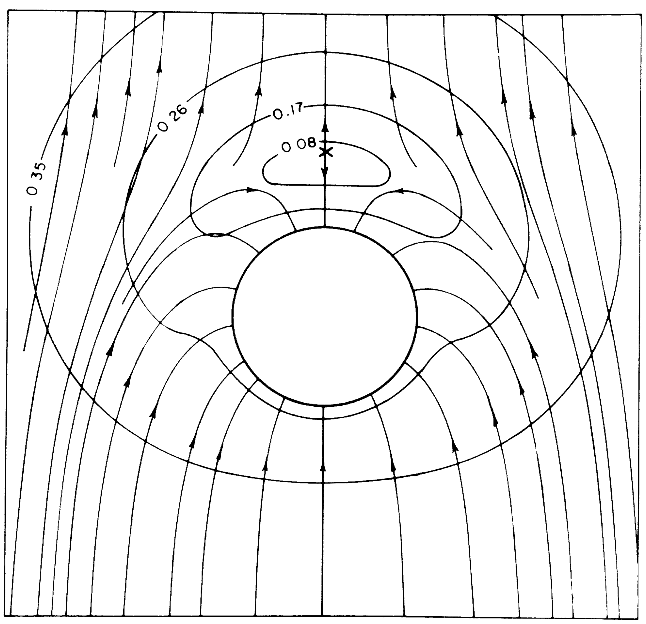

FIG. 1. Plot of the three-velocity field v^i/u^0 for flow with $v_\infty=0.6$ ($u_\infty=0.75$) past a Schwarzschild black hole. The streamlines are labeled by arrows while the velocity contours are labeled by values of v (in units of c). The inner circle is the event horizon with radius $r=2M$; the outer grid is at radius $r=7M$. The cross marks the stagnation point downwind of the hole.

The value of n can be found from the normalization equation (28).

The matter flux, from Eq. (7), is

$$\dot{N} = 4\pi(r_+^2+a^2)n_\infty u^0_\infty \tag{33}$$

for *all* flow directions at infinity. This is just n_∞ times the area of the black hole times the Lorentz γ factor for the asymptotic flow.

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[1]See, e.g., S. L. Shapiro and S. A. Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars: the Physics of Compact Objects* (Wiley, New York, 1983).

[2]H. Bondi and F. Hoyle, Mon. Not. Roy. Astron. Soc. **104**, 272 (1944).

[3]H. Bondi, Mon. Not. Roy. Astron. Soc. **112**, 195 (1952).

[4]F. C. Michel, Astrophys. Space Sci. **15**, 153 (1972).

[5]See, e.g., L. D. Landau E. M. Lifshitz, *Fluid Mechanics* (Addison-Wesley, Reading, MA, 1959), p. 504; or V. Moncrief, Astrophys. J. **235**, 1038 (1980).

[6]Moncrief, Ref. 5.

[7]See, e.g., C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Eq. (33.35).

[8]From *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. Stegun (U.S. GPO, Washington, D. C., 1965), Eq. (8.1.2).