Factorization of a Two-Loop Four-Point Superstring Amplitude

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The behavior of the two-loop four-point gravitation amplitude is investigated in the type-II superstring theory, where the genus-two Riemann surface degenerates into two tori. The amplitude turns out to be finite in this limit and it is shown explicitly to the lowest order of the pinching parameter that it produces the product of two one-loop amplitudes, as is expected from the factorization property.

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There has been much interest in superstring theories[1] as candidates for a unified theory of all interactions. Although superstring theories are expected to give us finite loop amplitudes to any order in perturbation theory, it is difficult to show the finiteness of amplitudes explicitly beyond the one-loop level, mainly because it is not yet clear how to treat so-called supermoduli for higher-genus Riemann surfaces. The purpose of this note is to clarify the role of supermoduli by looking at the factorization of a two-loop four-point superstring amplitude.

Recently it has been found that the picture-changing prescription[2] for supermoduli has some strange features, such as total derivative terms or the unphysical poles[3] of space-time supersymmetry current. The behavior of the degeneration limit of the amplitude seems to vary[4] depending on which part of the Riemann surface we place the picture-changing operators on. It turns out[5] that it is convenient to put the ghost field ξ suitably and to deform the contour of the Becchi-Rouet-Stora-Tyutin (BRST) current[2] in such a way that the degenerating Riemann surface has a correct physical projection at the two punctures. Here I use the prescription for supermoduli, where the positions of the supercurrents are integrated over the entire Riemann surface. In the degeneration limit, it is postulated that two of the super-Beltrami differentials become those corresponding to punctures and my prescription turns out to satisfy the correct factorization property. The relation between the present result and that by the picture-changing prescription will be briefly mentioned later.

Throughout this paper I restrict my analysis to the four-point graviton amplitude where the trivial homology cycle of the genus-two Reimann surface is pinched, and I discuss the degeneration only to the lowest order of the pinching parameter for simplicity. Although only the type-II superstring is considered here, the behavior of the amplitude of the (uncompactified) heterotic string theory[6] is the same, since if the tachyon pole is absent for right movers, then the amplitude behaves like $\int t^{-2} d^2 t$, which vanishes after the phase integral, as is the case at one loop (t is the pinching parameter).

The two-loop four-point graviton amplitude of the type-II superstring theory after integration over supermoduli is

$$A(1,2,3,4) = \prod_{j=1}^{4} \int d^2 u_j \prod_{i=1}^{2} \int d^2 z_i \int d^2 \tilde{z}_i \int \frac{\prod_{i \le j} d^2 \tau_{ij}}{(\det \operatorname{Im} \tau)^5} \sum_{\mathbf{m},\tilde{\mathbf{m}}:\text{even}} \det \mu_{i,\mathbf{m}}^{(3/2)}(z_j) [\det \mu_{i,\tilde{\mathbf{m}}}^{(3/2)}(\tilde{z}_j)]^* \times \langle T_F(z_1) T_F(z_2) T_F^*(\tilde{z}_1) T_F^*(\tilde{z}_2) V(u_1) V(u_2) V(u_3) V(u_4) \rangle_{\mathbf{m},\tilde{\mathbf{m}}} \times \theta_{\mathbf{m}}(0)^5 (Z_{3/2})_{\mathbf{m}}^{-1} \theta_{\tilde{\mathbf{m}}}^*(0)^5 (Z_{3/2}^*)_{\tilde{\mathbf{m}}}^{-1} | Z_2 Z_1^{-5} |^2, \quad (1)$$

where

$$V(u_j) = \zeta_{\mu\nu} (\partial X^\mu + ik \cdot \psi \psi^\mu)(\bar{\partial} X^\nu + ik \cdot \psi \psi^\nu) e^{ik_j \cdot X(u_j)}$$

is the graviton emission vertex, and the factors Z_λ are defined by $Z_\lambda = (\det' \bar{\partial}_1)^{1/2} \det' \bar{\partial}_\lambda$ for the reparametrization ghost ($\lambda = 2$), the superconformal ghost ($\lambda = \frac{3}{2}$), and scalar fields ($\lambda = 1$) (see Alvarez-Gaumé *et al.*,[7] Knizhnik,[7] Eguchi, and Ooguri,[7] and Dugan and Sonoda,[7] and Verlinde and Verlinde[8] for details).

$$T_F(z)_{z\theta} = \psi_\theta^\mu (\partial_z X^\mu + i\lambda_z^\mu) + (2c\, \partial\beta + 3\, \partial c \beta - \gamma b)_{z\theta},$$

$$T_F^*(\bar{z})_{z^*\theta^*} = \bar{\psi}_{\theta^*}^\mu (\bar{\partial}_{z^*} X^\mu + i\lambda_{z^*}^\mu) + (2c^*\, \bar{\partial}\beta^* + 3\, \bar{\partial} c^* \beta^* - \gamma^* b^*)_{z^*\theta^*}$$

are the supercurrents. $b,c,b^*,c^*,\beta,\gamma,\beta^*,\gamma^*$ are the reparametrization and the superconformal ghosts, respectively, and $\lambda_z^\mu, \lambda_{z^*}^\mu$ are the Lagrange multipliers introduced to make the term $\bar{\chi}^a \psi^\mu \bar{\chi}_a \psi_\mu$ linear in the gravitino χ^a in $d=2$ supergravity.[9] Note that there is no δ function in the correlation function $\langle (\partial_z X^\mu + i\lambda_z^\mu)(\bar{\partial}_{w^*} X^\nu + i\lambda_{w^*}^\nu) \rangle$ due to λ_a^μ. τ_{ij} is the period matrix of the genus-two Riemann surface and $\theta_{\mathbf{m}}(z)$ is the Riemann θ function[10] with spin structure

$\mathbf{m}=[\mathbf{m}_1,\mathbf{m}_2]$. $\mu_{i,\mathbf{m}}^{(3/2)}$ $(i=1,\ 2)$ is the super-Beltrami differential with conformal dimensions $(-\frac{1}{2},1)$. The coefficient, which appears in the summation over the spin structures, is identically 1 for flat ten-dimensional space-time. The reason is that one automatically gets the coefficient of the Gliozzi-Scherk-Olive (GSO) projection[11] for each torus from $(Z_{3/2})_{\mathbf{m}}$ in the degeneration limit, as can be seen below.

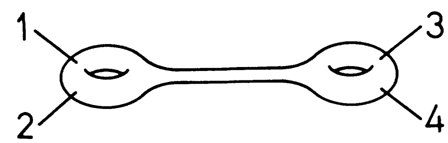

FIG. 1. The factorized two-loop diagram with two external lines on one torus and two on another.

Here I only discuss the degeneration limit as is shown in Fig. 1 for simplicity. Before analysis of the correlation functions, let us briefly look at the factor Z_λ in Eq. (1). In the degeneration limit $\tau\to\mathrm{diag}(\tau_1,\tau_2)$ (i.e., $t\equiv\tau_{12}\to 0$), where τ_i is the moduli parameter for each torus, two of the three Beltrami differentials and two $\mu^{(3/2)}$'s correspond to ordinary and super-Beltrami differentials for punctures on each torus, and these can be written as $\bar\partial$ derivatives of vector fields or the square root of vector fields[12] (see Refs. 8, 10, and 12, and Masur,[13] Yamada,[13] Wolpert,[13] and Nelson[13] for discussions on degeneration of Riemann surfaces). The integrations over the positions of these Beltrami differentials thus become contour integrals of the distributional Beltrami differentials around the annulus. Using the property of the contour integral around the annulus

$$\frac{1}{(2\pi i)^2}\oint dz_1\oint dz_2\, z_1^m z_2^n=\delta_{m,-1}\delta_{n,-1}, \tag{2}$$

we get, up to higher orders of the pinching parameter t and up to numerical constants,

$$t^2 Z_2\exp[2\pi i(\tau_1+\tau_2)]\sim Z_1\sim\exp[-(\pi/4)i(\tau_1+\tau_2)]\theta_1'(0\,|\,\tau_1)\theta_1'(0\,|\,\tau_2),$$

$$(Z_{3/2})_{\mathbf{m}}\sim t^{-1/2}\theta_{\mathbf{m}_1}(-2\Delta\,|\,\tau_1)\theta_{\mathbf{m}_2}(-2\Delta\,|\,\tau_2)=t^{-1/2}\exp[-\pi i(\tau_1+\tau_2)]C_{\mathbf{m}_1}C_{\mathbf{m}_2}\theta_{\mathbf{m}_1}(0\,|\,\tau_1)\theta_{\mathbf{m}_2}(0\,|\,\tau_2),$$

where I have used the fact that $\Delta=(1-\tau)/2$ for genus-one Riemann surfaces and $C_{\mathbf{m}_i}=\exp[2\pi i(m_i'-m_i'')]$ is the coefficient for the GSO projection for each torus. Putting determinants together, we get the following factorized form:

$$Z_2 Z_1^{-5}\theta_{\mathbf{m}}(0)^5(Z_{3/2})_{\mathbf{m}}^{-1}\sim t^{-3/2}\left(C_{\mathbf{m}_1}\frac{\theta_{\mathbf{m}_1}(0\,|\,\tau_1)^4}{\theta_1'(0\,|\,\tau_1)^4}\right)\left(C_{\mathbf{m}_2}\frac{\theta_{\mathbf{m}_2}(0\,|\,\tau_2)^4}{\theta_1'(0\,|\,\tau_2)^4}\right).$$

Let us now turn to correlation functions of the supercurrents and the vertex operators. In the degeneration limit, only the terms bilinear in ψ's contribute because of the summation over the spin structures, and so I only consider these terms in the following. Also, to the leading order of t, the integrand of $z_i,\bar z_i$ has a support only on the annulus. Using the variational formulas in Ref. 8 and the formula (2), I find after the summation over the spin structures that the only dominant contribution comes from the following (the contribution from the ghost supercurrents turns out to be of higher order in t):

$$\sum_{\mathbf{m},\bar{\mathbf{m}}:\mathrm{even}}\left\langle :\partial X^\mu(z_1)\partial X^\nu(z_2)\bar\partial X^\rho(\bar z_1)\bar\partial X^\sigma(\bar z_2):\prod_{j=1}^{4}e^{ik_j\cdot X(\mu_j)}\right\rangle\zeta_1^{\nu_1\sigma_1}\zeta_2^{\nu_2\sigma_2}\zeta_3^{\nu_3\sigma_3}\zeta_4^{\nu_4\sigma_4}$$

$$\times\theta_{\mathbf{m}}(0)^5(Z_{3/2})_{\mathbf{m}}^{-1}\langle :\psi^\mu(z_1)\psi^\nu(z_2):k_1\cdot\psi_1\psi_1^{\nu_1}k_2\cdot\psi_2\psi_2^{\nu_2}k_3\cdot\psi_3\psi_3^{\nu_3}k_4\cdot\psi_4\psi_4^{\nu_4}\rangle$$

$$\times\theta_{\bar{\mathbf{m}}}^*(0)^5(Z_{3/2}^*)_{\bar{\mathbf{m}}}^{-1}\langle :\bar\psi^\rho(\bar z_1)\bar\psi^\sigma(\bar z_2):k_1\cdot\bar\psi_1\bar\psi_1^{\sigma_1}k_2\cdot\bar\psi_2\bar\psi_2^{\sigma_2}k_3\cdot\bar\psi_3\bar\psi_3^{\sigma_3}k_4\cdot\bar\psi_4\bar\psi_4^{\sigma_4}\rangle$$

$$=\mathrm{const}\times|t^2|^{2-s/8}\zeta_1^{\mu\alpha}\zeta_2^{\nu\beta}\zeta_3^{\kappa\gamma}\zeta_4^{\lambda\delta}K_{\mu\nu\kappa\lambda}K_{\alpha\beta\gamma\delta}\frac{z_1-z_2}{z_1^2 z_2^2}\left(\frac{\tilde z_1-\tilde z_2}{\tilde z_1^2\tilde z_2^2}\right)^*$$

$$\times\left|\sum_{\mathbf{m}_1:\mathrm{even}}C_{\mathbf{m}_1}\frac{\theta_{\mathbf{m}_1}(u_1-p_1)^2}{E_1(u_1,p_1)^2}\frac{\theta_{\mathbf{m}_1}(u_2-p_1)^2}{E_1(u_2,p_1)^2}\right|^2\left|\frac{G_1(u_1,p_1)G_1(u_2,p_1)}{G_1(u_1,u_2)}\right|^{s/8}$$

$$\times\left|\sum_{\mathbf{m}_2:\mathrm{even}}C_{\mathbf{m}_2}\frac{\theta_{\mathbf{m}_2}(u_3-p_2)^2}{E_2(u_3,p_2)^2}\frac{\theta_{\mathbf{m}_2}(u_4-p_2)^2}{E_2(u_4,p_2)^2}\right|^2\left|\frac{G_2(u_3,p_2)G_2(u_4,p_2)}{G_2(u_3,u_4)}\right|^{s/8},$$

where $E_i(z,w)=\theta_1(z-w\,|\,\tau_i)/\theta_1'(0\,|\,\tau_i)$ is the prime form,[11]

$$G_i(z,w)=|E_i(z,w)|^2\exp\{-2\pi[\mathrm{Im}(z-w)]^2/\mathrm{Im}\,\tau_i\}$$

is the scalar propagator for each torus, respectively, p_i is the position of punctures on each torus, and the kinematical factor K is defined by

$$\begin{aligned}\zeta_1^\mu\zeta_2^\nu\zeta_3^\kappa\zeta_4^\lambda K_{\mu\nu\kappa\lambda}=&[s^2(t-u)/4](\zeta_1\cdot\zeta_3\zeta_2\cdot\zeta_4-\zeta_1\cdot\zeta_4\zeta_2\cdot\zeta_3)\\&+\tfrac{1}{2}s\zeta_1\cdot\zeta_3[u(\zeta_2\cdot k_3\zeta_4\cdot k_1+\zeta_2\cdot k_4\zeta_4\cdot k_2)-t(\zeta_2\cdot k_4\zeta_4\cdot k_1+\zeta_2\cdot k_3\zeta_4\cdot k_2)]\\&+\tfrac{1}{2}s\zeta_1\cdot\zeta_4[t(\zeta_2\cdot k_4\zeta_3\cdot k_1+\zeta_2\cdot k_3\zeta_3\cdot k_2)-u(\zeta_2\cdot k_3\zeta_3\cdot k_1+\zeta_2\cdot k_4\zeta_3\cdot k_2)]\\&+\tfrac{1}{2}s\zeta_2\cdot\zeta_3[t(\zeta_1\cdot k_3\zeta_4\cdot k_2+\zeta_1\cdot k_4\zeta_4\cdot k_1)-u(\zeta_1\cdot k_4\zeta_4\cdot k_2+\zeta_1\cdot k_3\zeta_4\cdot k_1)]\\&+\tfrac{1}{2}s\zeta_2\cdot\zeta_4[u(\zeta_1\cdot k_4\zeta_3\cdot k_2+\zeta_1\cdot k_3\zeta_3\cdot k_1)-t(\zeta_1\cdot k_3\zeta_3\cdot k_2+\zeta_1\cdot k_4\zeta_3\cdot k_1)].\end{aligned}\tag{3}$$

Here $s=-2k_1\cdot k_2$, $t=-2k_1\cdot k_4$, $u=-2k_1\cdot k_3$ are the Mandelstam variables. Note that (3) has on-shell gauge invariance (i.e., invariance under $\zeta_i\to\zeta_i+k_i$) and is symmetric under the exchange $(1\leftrightarrow 2)$ or $(3\leftrightarrow 4)$ or $(1\leftrightarrow 3$ and $2\leftrightarrow 4)$. Terms like $\zeta_1\cdot k_i\zeta_2\cdot k_j\zeta_3\cdot k_k\zeta_4\cdot k_l$ do not appear because of these symmetries. Thus, again by the formula (2), the amplitude (1) becomes

$$\begin{aligned}A(1,2,3,4)=\text{const}\times\zeta_1^{\mu\alpha}\zeta_2^{\nu\beta}\zeta_3^{\kappa\gamma}\zeta_4^{\lambda\delta}K_{\mu\nu\kappa\lambda}K_{\alpha\beta\gamma\delta}\int d^2t\,|t^2|^{-s/8}\prod_{i=1}^{4}\int d^2u_i\int\frac{d^2\tau_1}{(\mathrm{Im}\tau_1)^5}\int\frac{d^2\tau_2}{(\mathrm{Im}\tau_2)^5}\\\times\left|\frac{G_1(u_1,p_1)G_1(u_2,p_1)}{G_1(u_1,u_2)}\right|^{s/8}\left|\frac{G_2(u_3,p_2)G_2(u_4,p_2)}{G_2(u_3,u_4)}\right|^{s/8}.\end{aligned}\tag{4}$$

As is obvious from Eq. (4) there are neither tachyon poles nor massless poles in the intermediate states, but (4) has massive state poles at $s/8=N(\geq 1)$.

Now I compare (4) with the product of two one-loop amplitudes. Since I am discussing only the lowest order of t, I have to consider the one-loop three-point amplitudes with two massless external lines and one first massive state line. In the type-II superstring there are three kinds of first massive states.[14] Because of the summation over the spin structures, the only one state which has nonvanishing three-point amplitude is the antisymmetric third-rank tensor with respect to both left and right movers. It is easy to calculate this amplitude and we get

$$A(1,2,3)=\prod_{i=1}^{2}\int d^2u_i\int\frac{d^2\tau}{(\mathrm{Im}\tau)^5}\left|\frac{G(u_1,u_3)G(u_2,u_3)}{G(u_1,u_2)}\right|^{s/8}\zeta_3^{\mu\nu\kappa}k_3^\lambda\zeta_3^{\alpha\beta\gamma}k_3^\delta k_1^{[\mu}\zeta_1^\nu k_2^\kappa\zeta_2^{\lambda]}k_1^{[\alpha}\zeta_1^\beta k_2^\gamma\zeta_2^{\delta]},$$

for $s/8=1$. Note that this has on-shell gauge invariance $(\zeta_{3,\mu\nu\kappa}\to\zeta_{3,\mu\nu\kappa}+k_{3[\mu}\Lambda_{\nu\kappa]})$. In order to get the amplitude of Fig. 1, we have to sum over the polarization tensors:

$$\textstyle\sum_\zeta\zeta_{\mu\nu\kappa}\zeta^{\alpha\beta\gamma}=\delta_\mu^{[\alpha}\delta_\nu^\beta\delta_\kappa^{\gamma]}+(\text{terms with }k_\mu).$$

Because of the on-shell gauge invariance mentioned above, terms with momentum do not contribute to the amplitude, and we find that $\sum_{\zeta_5}A(1,2,5)A(5,3,4)$ coincides with Eq. (4). Hence we conclude that the first massive state pole in Eq. (1) is indeed consistent with the factorization property.

Finally I consider the same amplitude by using the picture-changing prescription. It turns out that if we place the picture-changing operator as in Fig. 2(a), then we have the correct factorized amplitude. It can be shown,[5] at least to the lowest order of t, that only the matter supercurrent $\gamma\psi\cdot\partial X$ contributes in the BRST current j_{BRST}. So, to this order, we can deform the contour of j_{BRST} as shown in Fig. 2(b) without picking up total derivative terms. This contour integral of j_{BRST} plays exactly the same role as that of the distributional super-Beltrami differentials which I discussed above, and the amplitudes with the two prescriptions coincide with each other. Notice that in both prescriptions only matter supercurrents contribute to the leading order of t. In the

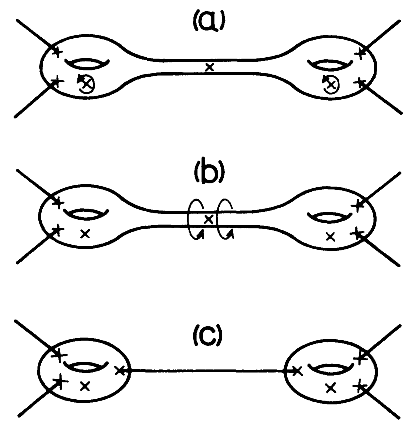

FIG. 2. (a) The picture-changing operator $\{Q,\xi\}$, which is defined as the contour integral of the BRST current around the ghost field ξ, is placed on each torus, and the zero mode ξ_0 is placed on the annulus. A cross stands for the ghost ξ. (b) The contours of the BRST currents on each torus are deformed onto the annulus. (c) In the degeneration limit, the two-loop diagram becomes the product of two one-loop diagrams and each torus has the picture-changing operators at punctures and the ghost zero mode ξ_0 at a certain point.

degeneration limit, Fig. 2(b) becomes Fig. 2(c), where the picture-changing operators guarantee that only physical states are propagating in the intermediate states. On the other hand, if two picture-changing operators are placed on the same torus, then it is not clear whether there is a physical projection in the intermediate states. Anyway, we see that supermoduli are important to have the physical poles in the intermediate states. Although I discussed only two-loop amplitudes here, if the nonrenormalization theorem from zero to three-point function can be proved for arbitrary genus, it is straightforward to generalize my analysis to any loop order.

Toward the completion of this work, I received a paper by Atick, Moore, and Sen[15] in which a global obstruction for super-Beltrami differentials is pointed out. Whether the present prescription satisfies their restriction is yet to be seen. The details and analysis of the amplitudes with other configurations of the external lines will be reported elsewhere.[5]

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