QCD Predictions for the Decay of the τ Lepton

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The semileptonic decay rate of a heavy lepton is shown to be rigorously calculable by use of perturbative QCD. Even the τ lepton is heavy enough for the perturbative approximation to be accurate. The prediction for the ratio $R=\Gamma(\tau\rightarrow\nu_\tau+\text{hadrons})/\Gamma(\tau\rightarrow\nu_\tau+e^-+\bar{\nu}_e)$ is 3.29, with an uncertainty of about 1% due to the uncertainty in the Λ parameter of QCD. Nonperturbative corrections to this prediction are estimated to be on the order of 1%. More importantly, these corrections are shown to be negative because of a fortuitous cancellation of the leading contribution from the gluon condensate. The resulting prediction is significantly smaller than the present experimental result $R=3.65\pm0.13$.

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The ratio $R(e^+e^-)=\sigma(e^+e^-\rightarrow\text{hadrons})/\sigma(e^+e^-\rightarrow\mu^+\mu^-)$ played an instrumental role in the development of QCD, because it provides evidence for color. Provided that the number of colors is $N_c=3$, a naive calculation of the cross section for producing quark-antiquark pairs gives a reasonable estimate for $R(e^+e^-)$, unless the center-of-mass energy is close to a resonance. Corrections to $R(e^+e^-)$ can be calculated systematically in perturbative QCD.

A corresponding ratio can be defined for decays of the τ lepton:

$$R=\frac{\Gamma(\tau^-\rightarrow\nu_\tau+\text{hadrons})}{\Gamma(\tau^-\rightarrow\nu_\tau+e^-+\bar{\nu}_e)}. \tag{1}$$

It is not well appreciated that this ratio also provides evidence for color.[1] A naive calculation of the decay rate of the τ^- into $\nu_\tau+d\bar{u}$ and $\nu_\tau+s\bar{u}$ gives $R=N_c$, where $N_c=3$ is the number of colors. This seems to be in reasonable agreement with the present experimental result[2] $R=3.65\pm0.13$. As will be shown in this Letter, corrections to the naive prediction for R for a heavy lepton can be calculated reliably by use of perturbative QCD. The integration over the energy of the neutrino makes the prediction insensitive to the effects of resonances and branch cuts. Nonperturbative corrections are shown to be negative because of the cancellation of the leading contribution from the gluon condensate. The resulting prediction for R is significantly smaller than the present experimental result.

Let me begin by expressing the ratio R as an integral over the invariant mass s of the hadrons:

$$R=\frac{2}{\pi}\int_0^{M^2}\frac{ds}{M^2}\left(1-\frac{s}{M^2}\right)^2\left[\left(1+2\frac{s}{M^2}\right)\text{Im}\Pi_T(s+i\epsilon)-\text{Im}\Pi_L(s+i\epsilon)\right], \tag{2}$$

where M is the mass of the heavy lepton. $\Pi_T(s)$ and $\Pi_L(s)$ are the transverse and longitudinal parts of the self-energy function of the W boson, with an overall factor of $e^2/96\pi^2\sin^2\theta_W$ removed for convenience. Their imaginary parts are proportional to their discontinuities across the positive real s axis, and therefore (2) can be expressed as an integral over a contour running from $s=M^2-i\epsilon$ to $s=0$ below the real axis and then back to $s=M^2+i\epsilon$ above the axis. Since Π_T and Π_L are analytic except on the positive real axis, this can be deformed into a contour which runs clockwise around the circle C of the radius $|s|=M^2$:

$$R=\frac{1}{i\pi}\int_C\frac{ds}{M^2}\left(1-\frac{s}{M^2}\right)^2\left[\left(1+2\frac{s}{M^2}\right)\Pi_T(s)-\Pi_L(s)\right]. \tag{3}$$

If M is sufficiently large, perturbative QCD can be used to calculate $\Pi_T(s)$ and $\Pi_L(s)$ for s near $-M^2$. The analytic continuations of these functions can then be used to approximate the integrand of (3) at the other points on the circle C. This method was in fact used by Lam and Yan[3] to predict the semileptonic decay rate of the τ lepton even before its discovery, and updated predictions have been given by Schilcher and Tran.[4]

The first contribution of this Letter is the observation that the perturbative prediction for R is in fact much more reliable than one might at first expect. The weakest point in the calculation seems to be that it uses the analytic continuations of the perturbative approximations to Π_T and Π_L even near the positive real s axis, where these functions have poles and branch cuts that cannot be described accurately by use of perturbative QCD. Fortunately, the integrand of (3) includes the factor $(1-s/M^2)^2$, which provides a double zero at $s=M^2$, effectively suppressing the contribution from the region near the branch cut. Even a pole due to a resonance would not contribute significantly to the integral.

This perturbative QCD prediction for the ratio R for a lepton of mass M will therefore be much more reliable than the calculation of $R(e^+e^-)$ at the center-of-mass energy M. The ratio $R(e^+e^-)$ is proportional to $\mathrm{Im}\Pi(M^2+i\epsilon)$, where $\Pi(s)$ is the photon self-energy function which can be calculated by use of perturbative QCD for s large and far from the positive real axis. To calculate $R(e^+e^-)$, the perturbative approximation to $\Pi(s)$ must be analytically continued to $s=M^2$. Perturbative QCD can tell us nothing about the accuracy of this analytic continuation. Indeed it is known to give a poor approximation if there are nearby resonances. In contrast, the calculation of the ratio R for a heavy lepton is insensitive to the accuracy of the analytic continuation near the positive real axis because of the suppression factor at $s=M^2$. Thus the perturbative QCD prediction for the ratio R for the τ lepton should be much more accurate than the prediction for $R(e^+e^-)$ at center-of-mass energies near M_τ.

To calculate the perturbative prediction for R using (3), we require analytic expressions for the perturbative approximations to the W boson self-energy functions. The renormalization-group equations can be used to expand the perturbative approximations to Π_T and Π_L in inverse powers of $\lambda=\ln(-s/\Lambda^2)$, where $\Lambda=150\mp 50$ MeV is the QCD scale parameter in the modified minimal-subtraction ($\overline{\mathrm{MS}}$) scheme. If we include all terms of order α_s^2 and neglect quark masses, the resulting expressions are

$$\Pi_T^{\mathrm{pert}}(s)=3\left[A-\lambda-\frac{2}{\beta_0}\ln\lambda+\frac{4K}{\beta_0^2}\frac{1}{\lambda}-\frac{4\beta_1}{\beta_0^3}\frac{\ln\lambda+1}{\lambda}\right],\quad \Pi_L^{\mathrm{pert}}(s)=0, \tag{4}$$

where $\beta_0=(33-2n_f)/6$ and $\beta_1=(153-19n_f)/12$ are the first two coefficients of the QCD β function,[5] $K=1.986-0.115n_f$ is the coefficient of the order-α_s^2 correction to $R(e^+e^-)$,[6] n_f is the number of light quarks, and A is a constant whose value is not relevant to the present analysis.

Since the perturbative approximations given in (4) are analytic functions of s except along the positive real axis, the simplest way to evaluate the contour integral (3) is to collapse the contour down around the positive real s axis. The resulting integral is identical to (2), except that $\mathrm{Im}\Pi_L=0$ and $\mathrm{Im}\Pi_T$ is replaced by

$$\mathrm{Im}\Pi_T^{\mathrm{pert}}(s+i\epsilon)=3\pi\left[1+\frac{2}{\pi\beta_0}\arctan\frac{\pi}{\lambda}+\frac{4K}{\beta_0^2}\frac{1}{\lambda^2+\pi^2}-\frac{4\beta_1}{\beta_0^3}\frac{\ln(\lambda^2+\pi^2)+1-(\lambda/\pi)\arctan(\pi/\lambda)}{\lambda^2+\pi^2}\right]. \tag{5}$$

I do not claim that (5) gives a good approximation to the integrand of (2), but only that it gives a good approximation to the integral, provided that M is large enough to allow a perturbation expansion in $\alpha_s(M)$.

In the case of the τ lepton whose mass is $M_\tau=1784$ MeV, the resulting prediction for the ratio R is

$$R^{\mathrm{pert}}=3.29\pm 0.04, \tag{6}$$

where the error is due to the uncertainty in Λ. The corrections of order α_s and α_s^2 to the naive prediction $R=3$ are 0.33 and -0.04, respectively, and so the perturbation expansion is well behaved. The perturbative prediction (6) differs by three standard deviations from the present experimental value for this ratio, which is $R=3.65\pm 0.13$. This value is obtained by the insertion of $B_e=(17.7\pm 0.4)\%$, the present world average[2] for the branching fraction of the τ into electrons, into the formula $R=(1-2B_e)/B_e$.

If there are any other sequential leptons heavier than the τ, perturbative QCD should provide very accurate predictions for their semileptonic decay rates. However, the mass of the τ lepton is sufficiently small that one should also consider the effects of nonperturbative corrections. For large negative s, the functions $\Pi_T(s)$ and $\Pi_L(s)$ can be expanded with the operator product expansion.[7] The leading nonperturbative corrections arise from the lowest-dimension operators which can develop nonzero vacuum expectation values. For the functions Π_T and Π_L, the leading corrections are

$$\begin{aligned}\Pi_T^{\mathrm{nonpert}}(s)&=3\left[\frac{\pi^2}{3}\frac{\alpha_s}{\pi}\langle GG\rangle\frac{1}{s^2}\left[1+\frac{7}{3\beta_0}\frac{1}{\lambda}\right]-\frac{256\pi^3}{81}\alpha_s\langle\bar\psi\psi\bar\psi\psi\rangle\frac{1}{s^3}\left[1+\frac{9}{4\beta_0}\frac{1}{\lambda}\right]\right],\\ \Pi_L^{\mathrm{nonpert}}(s)&=3\left[-8\pi^2 m\langle\bar\psi\psi\rangle\frac{1}{s^2}\right],\end{aligned} \tag{7}$$

where $m=(m_u+0.95m_d+0.05m_s)/2$ is a weighted average of quark masses. I have included the order-α_s corrections to the coefficient functions for the gluon condensate and the four-quark condensate,[8,9] and I have ignored the small anomalous dimension of the latter. The condensate values[7] are $(\alpha_s/\pi)\langle GG\rangle=(330\ \mathrm{MeV})^4$, $\alpha_s\langle\bar\psi\psi\bar\psi\psi\rangle=(240\ \mathrm{MeV})^6$, and $m\langle\bar\psi\psi\rangle=-(110\ \mathrm{MeV})^4$, with uncertainties of about a factor of 2.

Since the lowest-order terms of the coefficient functions appearing in (7) are analytic functions, their contributions to the integral (3) can be evaluated with use of Cauchy's residue theorem. This reveals that the only such terms that can contribute to R are the $1/s^3$ and $1/s^4$ terms in Π_T and the $1/s^2$ and $1/s^3$ terms in Π_L. In particular there is no leading-order contribution from the gluon condensate, since it contributes only to Π_T and its coefficient function is proportional to $1/s^2$. This fortuitous cancellation decreases the contribution from the gluon condensate by an order of magnitude. Its first contribution arises from the order-α_s correction to its coefficient function given in (7), which when expressed in analytic form contains the factor $1/\ln(-s/M^2)$. The leading nonperturbative contributions to R for the τ lepton are found to be

$$R^{\text{nonpert}}=0.002\frac{(\alpha_s/\pi)\langle GG\rangle}{(330\text{ MeV})^4}-0.015\frac{\alpha_s\langle\bar\psi\psi\bar\psi\psi\rangle}{(240\text{ MeV})^6}-0.013\frac{m\langle\bar\psi\psi\rangle}{(110\text{ MeV})^4}. \tag{8}$$

Taking into account the factor-of-2 uncertainties in the values of the condensates, we obtain the estimate $R^{\text{nonpert}}=-0.03^{+0.02}_{-0.03}$. These nonperturbative contributions are small, but more importantly, they are negative. This sign is a consequence of the strong suppression of the contribution from the gluon condensate and it is therefore insensitive to the precise values of the condensate parameters. Thus the nonperturbative corrections can only decrease the perturbative prediction (6). This would increase the discrepancy with the experimental value obtained from the measured branching fraction B_e.

The moments of the invariant-mass distribution of the hadrons produced by the decay of a heavy lepton can also be calculated by use of perturbative QCD. For example, the average invariant mass $\langle s\rangle$ is given by an expression analogous to (3):

$$R\frac{\langle s\rangle}{M^2}=\frac{1}{i\pi}\int_C\frac{ds}{M^2}\frac{s}{M^2}\left[1-\frac{s}{M^2}\right]^2\left[\left[1+2\frac{s}{M^2}\right]\Pi_T(s)-\Pi_L(s)\right]. \tag{9}$$

For τ decays, the lowest-order QCD prediction is $R\langle s\rangle/M_\tau^2=9/10$. When we include the order-α_s and order-α_s^2 corrections, the perturbative prediction becomes

$$(R\langle s\rangle/M_\tau^2)^{\text{pert}}=0.975\mp0.010, \tag{10}$$

where the error is due to the uncertainty in Λ. The leading nonperturbative contributions to $\langle s\rangle$ are

$$\left[R\frac{\langle s\rangle}{M_\tau^2}\right]^{\text{nonpert}}=-0.020\frac{(\alpha_s/\pi)\langle GG\rangle}{(330\text{ MeV})^4}+0.002\frac{\alpha_s\langle\bar\psi\psi\bar\psi\psi\rangle}{(240\text{ MeV})^6}+0.007\frac{m\langle\bar\psi\psi\rangle}{(110\text{ MeV})^4}. \tag{11}$$

In contrast to R, the largest contribution comes from the gluon condensate. If we take into account the uncertainties in the condensate parameters, the contribution (11) is $-0.011^{+0.019}_{-0.025}$.

I have shown that the decay rate of a heavy lepton into hadrons can be calculated reliably by use of perturbative QCD. For the τ lepton, the prediction for the ratio R defined in (1) is $R=3.29$. The errors due to the uncertainty in Λ and to nonperturbative corrections from operator condensates were calculated to be on the order of 1%. Moreover, the sign of the nonperturbative correction was shown to be negative. The resulting QCD prediction is 3 standard deviations below the experimental result $R=3.68\pm0.13$ derived from the measured branching fraction B_e of the τ into electrons. However, the branching fraction B_e can also be determined from the lifetime of the muon and the measured lifetime of the τ, $(3.07\pm0.10)\times10^{-13}$ s. The result[2] is $B_e=(19.2\pm0.6)\%$, which leads to the ratio $R=3.21\pm0.16$, which is in much better agreement with the QCD prediction. The difference between these two experimental results for R is related to a long-standing puzzle in τ decays: The sum of the partial widths for the known exclusive channels falls about 7% short of the total width.[10] One of the possible resolutions of this puzzle is that the measured branching fraction into electrons is too small by several standard deviations.[2] If this is indeed the case, it would bring the measurement of the ratio R for the τ lepton into perfect agreement with the prediction of perturbative QCD.

Note added.—Marciano[11] has pointed out that the electroweak corrections to the ratio R are significant because they are enhanced by the logarithm $\ln(M_Z/M_\tau)$. This correction increases the above predictions by about 2%.

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[1]M. L. Perl, Annu. Rev. Nucl. Part. Sci. **30**, 299 (1980).

[2]M. L. Perl, SLAC Report No. SLAC-PUB-4385, 1987 (to be published).

[3]C. S. Lam and T. M. Yan, Phys. Rev. D **16**, 703 (1977).

[4]K. Schilcher and M. D. Tran, Phys. Rev. D **29**, 570 (1984).

[5]W. Caswell, Phys. Rev. Lett. **33**, 244 (1974); D. R. T. Jones, Nucl. Phys. **B75**, 531 (1974).

[6]K. G. Chetyrkin, A. L. Kataev, and F. V. Tkachov, Phys. Lett. **85B**, 277 (1977); W. Celmaster and R. J. Gonsalves,

Phys. Rev. D **21**, 3112 (1980).
[7]M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B147**, 385, 448, 519 (1979).
[8]K. G. Chetyrkin, S. G. Gorishny, and V. P. Spiridonov, Phys. Lett. **160B**, 149 (1985); G. T. Loladze, L. R. Surguladze, and F. V. Tkachov, Phys. Lett. **162B**, 363 (1985).
[9]L. V. Lanin, V. P. Spiridinov, and K. G. Chetyrkin, Yad. Fiz. **44**, 1372 (1986) [Sov. J. Nucl. Phys. **44**, 892 (1986)].
[10]T. N. Truong, Phys. Rev. D **30**, 1509 (1984); F. J. Gilman and J. M. Rhie, Phys. Rev. D **31**, 1066 (1985); F. J. Gilman, Phys. Rev. D **35**, 3541 (1987).
[11]W. J. Marciano, unpublished.