Fermion Mass Hierarchy from Radiative Corrections

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(Received 14 December 1987)

A scheme is proposed to explain the hierarchy of fermion masses. In the model presented, which has no horizontal symmetry, generations pick up masses one by one through radiative corrections as we go to higher orders in perturbation theory. An extra generation of isosinglet heavy fermions plays an important role in implementing the scheme.

PACS numbers: 12.15.Ff

One of the puzzles left unexplained by the standard model of electroweak interactions is the pattern of fermion masses. Fermions of the first generation are very light when compared to the electroweak scale. Masses of other generations are also relatively small. Radiative corrections can easily generate such small numbers. In addition, they may help to explain the observed hierarchy of these masses.[1] I propose a scheme to realize this idea. In the model implementing the scheme, generations pick up masses one by one through radiative corrections as we go to higher loop orders. I do not impose any horizontal symmetry. However, an important role is played by an extra generation of isosinglet heavy fermions.

To illustrate the idea, consider a mass matrix M (say, for the up-quark sector) which at the tree level is of the form $aa^\dagger$ where a is a column vector in the generation space S. The number of nonzero eigenvalues (or the rank) of a matrix can be determined by our counting its zero eigenvalues. The relevant equation is $Mx=0$ where x is a vector in S. The number of linearly independent solutions to this equation is $n-r$ where n is the number of generations or the order of M and r is its rank. At the tree level this reduces to $a^\dagger x=0$ which can be satisfied by $n-1$ linearly independent x's. Hence M has rank one at this level. This means that one of the generations has picked up mass. When we go to one-loop level, the mass matrix will receive some correction. Let us assume that this correction is of the form $bb^\dagger$, where b is also a vector in S. Now the number of zero eigenvalues is determined by $a(a^\dagger x)+b(b^\dagger x)=0$ which implies $a^\dagger x=b^\dagger x=0$. If a and b are linearly independent, then this is satisfied by $n-2$ independent x's. This shows that the mass matrix has rank two. Thus at this order two generations are massive. This is true even when there are one-loop corrections proportional to $ab^\dagger$ and $ba^\dagger$ as can be easily seen. If the correction to the mass-matrix is small then the second eigenvalue will be small. Other generations get masses one by one as this process is continued by our going to higher loop levels. Smaller and smaller eigenvalues get added. This can explain the observed hierarchy of fermion masses. A similar matrix structure was attempted by Baur and Fritzsch[2] to obtain the masses of composite quarks and leptons as electromagnetic self-energies.

Now I present a model which implements the above scheme. Let us work with left-right symmetry[3] where the gauge group is

$$SU(3)_C\otimes SU(2)_L\otimes SU(2)_R\otimes U(1)_{B-L}.$$

I analyze only the quark sector in this Letter. $SU(2)_L$ doublet quarks are denoted by Q_{Li} where i is the generation index. Similarly Q_{Ri} represents $SU(2)_R$ doublet quarks. They all have $B-L$ charges $\frac{1}{3}$. In addition, I include an extra generation of fermions which are singlets[4-7] under $SU(2)_L\otimes SU(2)_R$. They are denoted by P and N having $B-L$ charges $\frac{4}{3}$ and $-\frac{2}{3}$, respectively. In addition to Q's, P and N are also color triplets. The Higgs sector consists of an $SU(2)_L$ doublet χ_L and $SU(2)_R$ doublet χ_R each with $B-L$ charge 1. In addition, I use a scalar field ω which is a singlet under $SU(2)_L\otimes SU(2)_R$ with $B-L$ charge $-\frac{2}{3}$. It is taken to be a color triplet. One also needs a parity-odd singlet Higgs to break left-right symmetry.[7] However, it will not affect the results. With this set of fields we can write down the Yukawa couplings (along with the mass terms for P and N) as follows:

$$L_Y=\sum_{ij}H_{ij}(Q_{Li}^TC^{-1}\tau_2\omega Q_{Lj}+Q_{Ri}^TC^{-1}\tau_2\omega Q_{Rj})+\sum_i h_i^P(\bar{Q}_{Li}\tilde{\chi}_LP_R+\bar{Q}_{Ri}\tilde{\chi}_RP_L)$$
$$+\sum_i h_i^N(\bar{Q}_{Li}\chi_LN_R+\bar{Q}_{Ri}\chi_RN_L)+f(P_L^TC^{-1}\omega N_L+P_R^TC^{-1}\omega N_R)+M_P\bar{P}_LP_R+M_N\bar{N}_LN_R+\text{H.c.}, \quad (1)$$

where C is the charge-conjugation matrix, τ_2 is the SU(2) metric, and $\tilde{\chi}=i\tau_2\chi^*$. Parity conservation is assumed for simplicity. Color indices are suppressed. H_{ij} is found to be a symmetric matrix. ω, being charged, does not receive any vacuum expectation value. The vacuum expectation values for χ_L and χ_R are

$$\langle\chi_L\rangle=\begin{bmatrix}0\\ v_L\end{bmatrix},\quad \langle\chi_R\rangle=\begin{bmatrix}0\\ v_R\end{bmatrix}. \tag{2}$$

In the following the mass matrix is analyzed for the up sector only. The same can be carried out for the down sector. The mass terms for the up sector including P are, in general, of the form

$$\textstyle\sum_{ij}\delta M_{ij}\bar{u}_{Li}u_{Rj}+\sum_i v_L\alpha_i\bar{u}_{Li}P_R+\sum_j v_R\beta_j^*\bar{P}_Lu_{Rj}+M_P\bar{P}_LP_R+\text{H.c.}, \tag{3}$$

where u stands for the up sector of the quark doublet. This results in the following mass matrix:

$$M_T=\begin{bmatrix}\delta M_{ij} & v_L\alpha_i\\ v_R\beta_j^* & M_P\end{bmatrix}. \tag{4}$$

From (1) and (2) we find that at the tree level $\delta M=0$ and $\alpha=\beta=h^P$. δM comes purely from radiative corrections. We note that there are no counterterms in the bare Lagrangean to cancel any divergent contributions to δM at any order. Hence renormalizability implies that δM is finite. Other parameters in (4) also receive some corrections. The number of massive fermions in the up sector is given by the rank of $M_TM_T^\dagger$ or $M_T^\dagger M_T$. But $M_T^\dagger M_T$ and M_T have the same rank. This is because $M_T^\dagger M_Tx=0$, which gives the number of zero eigenvalues, implies $M_Tx=0$ and vice versa. To find the rank of M_T, consider the problem of counting its zero eigenvalues. The relevant set of equations is

$$\begin{aligned}&\textstyle\sum_j\delta M_{ij}x_j+v_L\alpha_ix_{n+1}=0,\\ &\textstyle\sum_j v_R\beta_j^*x_j+M_Px_{n+1}=0.\end{aligned} \tag{5}$$

Eliminating x_{n+1} from above, we get

$$\textstyle\sum_j(\delta M_{ij}+a_0\alpha_i\beta_j^*)x_j=0, \tag{6}$$

where $a_0=-v_Lv_R/M_P$. I will refer to the combination $\delta M+a_0\alpha\beta^\dagger$ as M. For M_P large compared to v_L and v_R, which I refer to as the seesaw limit, M coincides with the mass matrix for the up sector excluding P. If M has rank r and n is its order (for n ordinary generations) then there are $n-r$ linearly independent solutions to (6). This shows that M_T has $n-r$ zero eigenvalues and hence, being a matrix of order $n+1$, it has rank $r+1$. This means that there are $r+1$ nonzero eigenvalues to $M_TM_T^\dagger$. One of them corresponds to a heavy fermion representing P. Thus r generations are massive at this order. Our problem is reduced to finding the rank of M. At the tree level, we found that $\delta M=0$ and $\alpha=\beta=h^P$. Thus $a_0h^Ph^{P\dagger}$ is the tree-level contribution to M. In this case $r=1$. Only one ordinary generation has gained mass at this level. The corresponding mass eigenvalue, in the seesaw limit, has the value $a_0h^{P\dagger}h^P$ at the tree level. In the $n=3$ case, top and bottom quarks become massive at this order.

Our problem is to examine how r increases as we go to higher loop levels. At one-loop level, δM gets a contribution from the graph shown in Fig. 1(a) which is proportional to $H^\dagger h^{N*}(H^\dagger h^{N*})^\dagger$. There is another one-loop graph with χ_L and χ_R in the internal boson line. However, its contribution, being proportional to the tree-level value of M, will not change r. α and β also receive corrections proportional to $H^\dagger h^{N*}$. Following our discussion at the beginning of the Letter, we find that r becomes 2 at one-loop level if h^P and $H^\dagger h^{N*}$ are linearly independent. This ensures mass hierarchy for $n=3$. Then charm and strange quarks pick up masses at one-loop level while up and down quarks do so at two-loop order. Example of a two-loop graph is given in Fig. 1(b). We will find later that the dominant contributions to the charm and the up-quark masses are given by the graphs of Figs. 1(a) and 1(b), respectively. We note an important property of all the graphs contributing to δM. A quark line does not lose its dependence on the generation index by coupling to ω. However, to give a correction to the mass matrix, it has to get converted to P or N. At this transition the index dependence is lost. This has the effect of factorizing the correction into a form $ab^\dagger$. This property is responsible for producing a hierarchy of mass eigenvalues. If there is only one new vector intro-

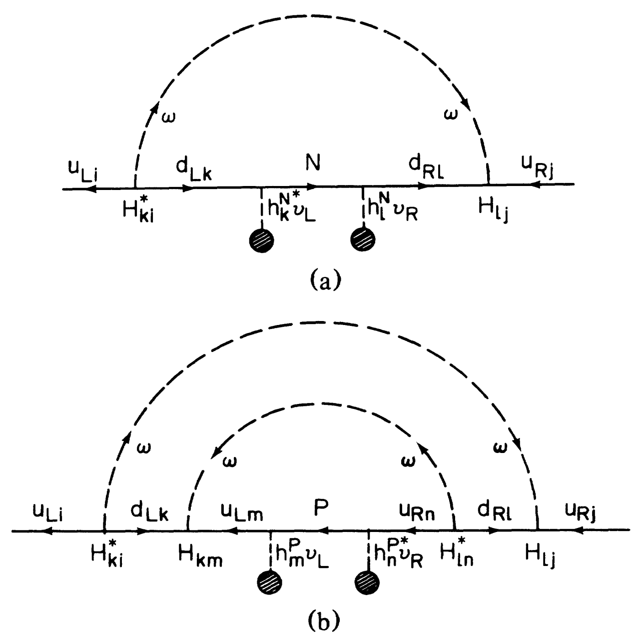

FIG. 1. Examples of one- and two-loop graphs contributing to the up-quark mass matrix.

duced at each order, mass hierarchy is ensured when all such vectors form a linearly independent set. But for $n>3$, one finds that two new vectors are introduced at two-loop level thereby incrementing the rank by two. However, the number of eigenvalues significant at that order increases only by one as can be shown by the following analysis.

Let us perform perturbative analysis on M to obtain the dominant contributions to its eigenvalues. I will show that these eigenvalues are ordered in successive powers of the loop expansion parameter (which is essentially $1/16\pi^2$ coming from loop integration). Collecting the radiative corrections from various loop levels, we get the following expansion for M:

$$M=M_0+\lambda M_1+\lambda^2 M_2+\cdots, \tag{7}$$

where λ keeps track of the loop orders and M_i contributes at ith-loop level. To find the eigenvalues of M, we have to solve the equation $M|m\rangle=m|m\rangle$ where $|m\rangle$ is an eigenvector of M with eigenvalue m. I use the bra and ket notation for vectors in the generation space. In analogy to (7), we assume the following expansions for m and $|m\rangle$:

$$\begin{aligned} m&=m_0+\lambda m_1+\lambda^2 m_2+\cdots,\\ |m\rangle&=|0\rangle+\lambda|1\rangle+\lambda^2|2\rangle+\cdots. \end{aligned} \tag{8}$$

I will show that there exists only one eigenvector with $m_0\neq 0$, only one with $m_0=0$, but $m_1\neq 0$, only one with $m_0=m_1=0$, but $m_2\neq 0$, and so on. Besides ensuring hierarchy, this will give us the dominant contributions to the eigenvalues. For this purpose, we substitute (7) and (8) into $M|m\rangle=m|m\rangle$ and collect the coefficients of each power of λ. We obtain a set of equations of which the first three are

$$(M_0-m_0)|0\rangle=0, \tag{9}$$

$$(M_0-m_0)|1\rangle+(M_1-m_1)|0\rangle=0, \tag{10}$$

$$(M_0-m_0)|2\rangle+(M_1-m_1)|1\rangle+(M_2-m_2)|0\rangle=0. \tag{11}$$

Before proceeding further, let us look for a general expression for M_i. M_0, being the tree-level value of M, is given by $a_0|h\rangle\langle h|$. I have dropped the superscript P which will be implicit at the relevant places from now on. Figure 1(a), contributing to δM at one-loop order, is proportional to $|Hh\rangle\langle Hh|$ where $Hh\equiv H^\dagger h^{N*}$. Figure 1(b), significant at two-loop level, is proportional to $|H^2h\rangle\langle H^2h|$ where $H^2h\equiv H^\dagger Hh^P$. Contributions from other graphs can be similarly found. There are also corrections to α and β. Looking at some of the graphs one can come up with a general expression for M_i:

$$M_i=\sum_{0\le k+l\le 2i} a_{i,kl}|H^k h\rangle\langle H^l h|. \tag{12}$$

H^k is a short form for $H^\dagger HH^\dagger H\cdots$ involving k matrices. In $H^k h$, $h=h^P$ if k is even and $h=h^{N*}$ otherwise. Some of the coefficients in (12) will be zero if there are no graphs contributing to them (for instance, $a_{1,20}=a_{1,02}=0$). $a_{1,11}$ and $a_{2,22}$ are obtained by the evaluation of the finite graphs shown in Figs. 1(a) and 1(b), respectively. In the following, we find that the dominant contributions to the first three eigenvalues of M are

$$a_0\langle h|h\rangle,\quad a_{1,11}\langle Hh|P_1|Hh\rangle,\quad a_{2,22}\langle H^2h|P_2|H^2h\rangle,$$

where P_1 and P_2 are projection operators satisfying $P_1|h\rangle=0$ and $P_2|h\rangle=P_2|Hh\rangle=0$. In order that the eigenvalues are not trivially zero, the set $\{h,Hh,\ldots,H^{n-1}h\}$ should be linearly independent.

Let us start with $m_0\neq 0$. Using $M_0=a_0|h\rangle\langle h|$ in (9), we find that $|0\rangle\propto|h\rangle$ is the only solution with this property. The corresponding eigenvalue is $m\simeq m_0=a_0\langle h|h\rangle$. Next let us consider the case $m_0=0$, but $m_1\neq 0$. Now (9) tells us that $\langle h|0\rangle=0$ implying $P_1|0\rangle=|0\rangle$. Multiplying (10) by P_1 and noting that $P_1M_0=0$, we get

$$P_1M_1P_1|0\rangle=m_1|0\rangle, \tag{13}$$

which is an eigenvalue equation for the matrix $P_1M_1P_1$. M_1 can be obtained from (12) with $i=1$. Then $P_1M_1P_1$ simplifies to $a_{1,11}P_1|Hh\rangle\langle Hh|P_1$. Using this in (13), we find that $|0\rangle\propto P_1|Hh\rangle$ is the only solution with $m_1\neq 0$. The corresponding eigenvalue is

$$m\simeq m_1=a_{1,11}\langle Hh|P_1|Hh\rangle.$$

Next we come to the case $m_0=m_1=0$, but $m_2\neq 0$. Again (9) implies that $\langle h|0\rangle=0$ leading to $P_1|0\rangle=|0\rangle$. Multiplying (10) by P_1, we get (13). However, since now $m_1=0$, (13) says that $\langle Hh|0\rangle=0$ leading to $P_2|0\rangle=|0\rangle$. Multiplying (11) by P_2 and noting that $P_2M_0=P_2M_1=0$, we find that m_2 is an eigenvalue of the matrix $P_2M_2P_2$. With M_2 obtained from (12) this matrix simplifies to $a_{2,22}P_2|H^2h\rangle\langle H^2h|P_2$. Following our earlier analysis, we find that only one vector with $|0\rangle\propto P_2|H^2h\rangle$ picks up a nonzero eigenvalue given by $m\simeq m_2=a_{2,22}\langle H^2h|P_2|H^2h\rangle$ at this order. What we have shown above is that for $n\le 3$, the number of significant eigenvalues of M increases by one at each order until it becomes n. One can prove this by induction for any n. The eigenvalues we have obtained are the mass eigenvalues in the seesaw limit. Perturbative analysis on $M_TM_T^\dagger$ will give us these masses in the general case.[8] For M_P, $v_R\gg v_L$, the tree-level mass eigenvalue is $(1+v_R^2\langle h|h\rangle/M_P^2)^{-1/2}a_0\langle h|h\rangle$. Other eigenvalues that we obtained in the seesaw limit are the dominant contributions to the masses in the general case as well.

The model as such cannot explain the hierarchy of mixing angles. This is because the matrices that diagonalize the up and down mass matrices are in general different at the tree level itself. However, one accounts for the observed mixing hierarchy by equating h^P and h^N with a softly broken discrete symmetry.[8] The result is that the mixings V_{us} and V_{cb} are of $O(\lambda)$ while V_{ub} is of $O(\lambda^2)$. This symmetry leads to an understanding of the isodoublet mass splittings as well.[8] From the seesaw-limit expression for the tree-level mass eigenvalue, one notes that $m_t \gg m_b$ requires $M_N \gg M_P$. Since N and P quarks contribute respectively to the up and down sectors at one loop and vice versa at two loops (as can be seen from Fig. 1), we naturally obtain $m_c > m_s$ and $m_u < m_d$ when mass of ω is of the order of M_N. Up to now our discussion was confined to the quark sector. Mass hierarchy in the charged-lepton sector will follow from a similar analysis. One introduces a scalar η and two heavy leptons N^0 and E to play the role of ω, P, and N, respectively. However, to explain the lightness of the neutrinos, one needs to invoke the seesaw mechanism.[9] If the heavy neutrino N^0 has a Majorana mass term, one of the right-handed neutrinos picks up a heavy Majorana mass at the tree level[5] while others do so from radiative corrections thereby leading to light neutrinos in the observed sector.[10] The ordering that was assumed among the various mass scales v_L, v_R, M_P, M_N, and M_ω needs a natural explanation which is absent in my model. The masses of the isosinglets are not fixed relative to v_L or v_R since they are not protected by any symmetry. These topics need further study.

In conclusion, I note that radiative corrections can explain the hierarchy of fermion masses. A matrix H, given by a coupling of quarks to a field that does not receive any vacuum expectation value, and a vector h, responsible for one ordinary generation to pick up mass at the tree level, are the main ingredients of my scheme. It is of interest to look for other possible implementations of this mechanism.

It is a pleasure to thank Professor R. N. Mohapatra for his valuable guidance during this work. I would like to thank Alex Kagan for discussions. This work was supported in part by a grant from the National Science Foundation.

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