Where the Sign of the Metric Makes a Difference

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The groups Pin(n,m) and Pin(m,n) are not isomorphic, and the obstruction classes to their respective bundles are different. It follows that for nonorientable superstring theories, the contributions to a Polyakov path integral from surfaces with positive metric are different from the contributions from those with negative metrics.

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Since Dirac, physicists have known how to define spinors in flat space-time. Spinors transform under the group Spin(m,n), the double cover of the appropriate rotation or Lorentz group SO(m,n), and Spin groups can be constructed explicitly from the Clifford algebra of the γ matrices.

For general relativity and for superstring theory, however, one must consider spinors in curved backgrounds. The questions here are more subtle. The flat-space construction still applies locally, but there can be a topological obstruction to its global extension: It is not always possible to ensure that locally defined spinors match up consistently between coordinate patches. For orientable manifolds, the obstruction is known as the second Stiefel-Whitney class, w_2; on manifolds for which $w_2 \neq 0$, ordinary spinors do not exist.

For nonorientable manifolds, even the local construction is more difficult. One must now start with the double cover Pin(n,m) of O(m,n), since the reduction of O(n,m) to SO(n,m) can no longer be defined globally.[1] The topological obstruction to the existence of spinors now depends on the choice of m and n. Surprisingly, this means that the overall sign of the metric is important.

At first sight it would not seem to make any physical difference whether one chooses a metric g_{AB} with signature (n plusses, m minuses) or (m plusses, n minuses). In particular, the groups O(n,m) and O(m,n) which leave invariant the quadratic forms defined respectively by metrics of signatures (n,m) and (m,n) are isomorphic. So are the double coverings Spin(n,m) and Spin(m,n) of SO(n,m) and SO(m,n), respectively. However, the double coverings Pin(n,m) and Pin(m,n) of O(n,m) and O(m,n) are not isomorphic. This is not unexpected, since the corresponding Clifford algebras $\mathcal{C}(n,m)$ and $\mathcal{C}(m,n)$ are not isomorphic. A simple example will illustrate this point. Pin$(0,1) \equiv$ Pin$^+(1)$ consists of four elements $\pm\mathbb{1}, \pm\gamma$ such that

$$\gamma^2 = \mathbb{1};$$

hence,

$$\mathrm{Pin}^+(1) \sim \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Pin$(1,0) \equiv$ Pin$^-(1)$ consists of four elements $\pm\mathbb{1}, \pm\gamma$ such that

$$\gamma^2 = -\mathbb{1};$$

hence,

$$\mathrm{Pin}^-(1) \sim \mathbb{Z}_4.$$

Our convention is

$$\gamma_A\gamma_B + \gamma_B\gamma_A = -2g_{AB}\mathbb{1}. \tag{1}$$

If we had used

$$\gamma_A\gamma_B + \gamma_B\gamma_A = 2g_{AB}\mathbb{1}, \tag{2}$$

the results would have been interchanged but the issue would remain the same. It is, in general, useful to label a Clifford algebra $\mathcal{C}(Q)$ by the quadratic form Q, which reflects the combined choice of (n,m) vs (m,n) and Eq. (1) vs Eq. (2). However, in the discussion below of superstrings, it will be convenient to treat these two choices separately.

Since the groups Pin$^+(n)$ and Pin$^-(n)$ are different, the obstruction classes for Pin$^+$ bundles and Pin$^-$ bundles are likely to be different. Indeed, Karoubi[2] has shown[3] that their obstruction classes are respectively

$$w_2(\mathrm{Pin}^+(n))$$

and

$$w_2(\mathrm{Pin}^-(n)) + w_1(\mathrm{Pin}^-(n)) \cup w_1(\mathrm{Pin}^-(n)),$$

where w_1 and w_2 are the first and second Stiefel-Whitney classes.[4] The obstruction class for a Pin(m,n) bundle with m and n both nonvanishing is more complicated and will not be discussed further here.

The first Stiefel-Whitney class is the obstruction class to orientability. A manifold is orientable if and only if the first Stiefel-Whitney class of its frame bundle is trivial; one then writes $w_1 = 0$. The first Stiefel-Whitney class can be constructed explicitly from the determinants of a family of transition functions of the bundle. The second Stiefel-Whitney class can be constructed explicitly from the lifts of these transition functions to a Pin(n,m) bundle.

Simple statements can be made about the obstruction to $\mathrm{Pin}^{\pm}(2)$ bundles over two-dimensional closed surfaces without boundary. For such surfaces, w_2 and $w_1 \cup w_1$ are both equal to the Euler characteristic χ mod 2 of the surface, and so

$$w_2 + w_1 \cup w_1 = 2\chi \bmod 2 = 0,$$

and one can always construct a $\mathrm{Pin}^-(2)$ structure. The second Stiefel-Whitney class vanishes if and only if χ is even. The Euler characteristic χ is always even if the surface is orientable. Hence, an obstruction to a $\mathrm{Pin}^+(2)$ structure can exist only for a nonorientable surface. Any nonorientable surface can be constructed either from the Klein bottle or from the real projective two-plane via connected sums with orientable surfaces. To determine the closed nonorientable surfaces which do not admit a $\mathrm{Pin}^+(2)$ structure it is sufficient to refer to the following table.

	$\mathbb{R}P(2)$	Klein bottle
w_1	$\neq 0$	$\neq 0$
$w_1 \cup w_1$	1	0
w_2	1	0

The Euler characteristic of the connected sum of two surfaces is

$$\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2.$$

Thus, the connected sum of any orientable closed two-surface with $\mathbb{R}P(2)$ does not admit a $\mathrm{Pin}^+(2)$ structure.

This result has important implications for string theory. The Polyakov path integral for the Neveu-Schwarz-Ramond superstring is ordinarily defined as an integral over two surfaces with positive definite metrics, with convention (2) for the Clifford algebra. These choices imply that the relevant Pin group is $\mathrm{Pin}^+(2)$. Hence, pinors cannot be constructed over all surfaces, and the theory is undefined on connected sums of oriented closed two surfaces with $\mathbb{R}P(2)$. Our results suggest that one should start instead with surfaces with negative definite metrics, for which $\mathrm{Pin}^-(2)$ structures always exist. For orientable surfaces, $w_1 \cup w_1 = 0$, and $\mathrm{Pin}^{\pm}$ bundles are reducible to a unique spin bundle; hence, for such surfaces, the theory with negative definite metrics is equivalent to the standard theory. For nonorientable surfaces, however, the two versions are inequivalent. We are thus led to the surprising conclusion that the overall sign of the metric makes a difference for string theory.

Rather than changing the sign of the metric, one might instead change the Clifford algebra convention from (2) to (1), but the resulting theory would no longer be supersymmetric. Indeed, there are several sign choices to be made in the choice of a superstring Lagrangean, but not all are independent. If we set $\epsilon_i = \pm 1$, the Lagrangean may be written

$$\mathcal{L} = -\tfrac{1}{2}\epsilon_1 g^{AB}\partial_A X \partial_B X - \tfrac{1}{2} i \bar{\Psi}\gamma^A \partial_A \Psi + \text{other terms},$$

where

$$\{\gamma_A, \gamma_B\} = 2\epsilon_2 g_{AB}$$

and

$$\bar{\Psi} = \Psi C \quad \text{with } C = \epsilon_3 C^{\mathrm{T}}.$$

Given the signature of the metric, the quantity ϵ_3 is determined uniquely by the requirement that Majorana pinors exist locally.[5] Further, the Lagrangean $\mathcal{L}$ is invariant under the global supersymmetry transformation

$$\delta X^\mu = \bar{\alpha}\Psi^\mu, \quad \delta\Psi^\mu = i\gamma^A \partial_A X^\mu \alpha$$

if and only if

$$\epsilon_2 = \epsilon_1 \epsilon_3.$$

Hence, if the Clifford-algebra convention (the sign of ϵ_2) is changed, the sign of ϵ_1 must also be. The combined effect of these changes of ϵ_2 and ϵ_1 is the same as that of simply changing the sign of the metric.

Grinstein and Rohm[6] have discovered a related difficulty in constructing pinors and have concluded that the Polyakov path integral for type-I superstrings is inconsistent as it stands. They consider only what we call $\mathrm{Pin}^+(2)$ bundles; but in an addendum (to appear) they note that a different choice of parity would have been equivalent to choosing a different Pin(2) group.

The difference between Pin^+ and Pin^- has been noted in a different context by Dabrowski and Percacci[7] in a preprint which has recently come to our attention. They have computed the transformations of $\mathrm{Pin}^+(2)$ and $\mathrm{Pin}^-(2)$ structures under the action of diffeomorphisms of their base manifolds not isotopic to the identity. Their obstruction criterion for $\mathrm{Pin}^+(2)$ structures is given in terms of the genus of the base manifold. It is identical to ours if one interprets[8] the "genus of a nonorientable surface" as the genus of its orientable double covering.

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[1]The words Pin and pinor are to Spin and spinor as O(n) is to SO(N). This terminology is convenient for distinguishing the double covers of O(n) and SO(n). M. F. Atiyah, R. Bott, and A. Shapiro, Topology Suppl. 1, **3**, 3–38 (1963).

[2]Max Karoubi, Ann. Sci. Ecole Norm. Sup. **4**, 161–270 (1968).

[3]For a more intuitive proof using explicit constructions of the Stiefel-Whitney classes in terms of transition functions of the bundles, see Yvonne Choquet-Bruhat and Cécile DeWitt-Morette, in Analysis, Manifolds and Physics, edited by Y. Choquet-Bruhat and C. DeWitt-Morette (North-Holland, Amsterdam, to be published), Pt. 2.

[4]For the definition of Stiefel-Whitney classes, see, for instance, John W. Milnor and James D. Stasheff, *Characteristic Classes* (Princeton Univ. Press, Princeton, NJ, 1974).

[5]Peter van Nieuwenhuizen, in *Relativity, Groups and Topology II,* edited by B. S. DeWitt and R. Stora (North-Holland, Amsterdam, 1984), p. 850.

[6]Benjamin Grinstein and Ryan Rohm, Commun. Math. Phys. **111**, 667–675 (1987).

[7]L. Dabrowski and R. Percacci, International School for Advanced Studies Report No. 103/86/EP, 1986 (to be published).

[8]W. S. Massey, *Algebraic Topology, An Introduction* (Springer-Verlag, Berlin, 1967), p. 33.