Nonequilibrium Statistical Mechanics Model Showing Self-Sustained Oscillations

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In a simple nonequilibrium statistical mechanics model (two coupled nonlinear stochastic reaction-diffusion equations), self-sustained oscillations appear as a symmetry-breaking effect for the associated probability density. Correlation and response functions are approximately calculated above and below the transition point. Renormalization-group theory yields the leading-order scaling of order parameter, relaxation time, correction to the collective frequency, and correlation length.

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Collective temporal oscillations appear in many systems of physical interest.[1] Although limited to mean-field models so far, a statistical mechanical theory of systems which undergo phase transitions from quiescent to time-periodic phases has come to the fore in recent years: Scheutzow[2] studied a stochastic Brusselator reaction-diffusion model, Shiino[3] considered a problem of self-synchronization of nonlinear oscillators with an external noise, and this author[4] analyzed a model of sliding charge-density waves. In all three cases, the physical system is typically composed of infinitely many interacting nonlinear subsystems, oftentimes in contact with a thermal bath or with a source of external noise. Each subsystem is able to oscillate when disconnected from the others and from the noise source. When noise and interaction are present, the subsystems may synchronize their oscillations and produce a stable state of collective rhythmicity.

Within the limitation to mean-field interactions, the onset of the stable collective oscillations is entirely analogous to an equilibrium phase transition: In an infinite system, at a critical value of the temperature or another control parameter, the stationary density bifurcates, thereby ceasing to be unique and stable. The amplitude of the stable bifurcating solution may be measured by an order parameter which is the average of some quantity. Other "nonequilibrium phase transitions" or noise-induced phase transitions studied in the literature[5] are different: Usually they arise in finite systems whose unique probability density changes from having one peak to having several ones. Different maxima are supposed to represent distinct phases.

While the deterministic aspect of temporal oscillations are well understood, this is not true in the general stochastic case when there is a short-range interaction between different subsystems. This is somewhat surprising given the extensive literature on dynamical critical phenomena, which, after all, deals with stochastic nonlinear reaction-diffusion (Landau-Ginzburg) equations.[6] In this literature, the dynamical aspects usually refer to relaxation toward stationary states of the probability law associated with the equations, or to responses to weak time-dependent external fields. A stable time-dependent probability law (persisting in the long-time limit) is neither searched for nor found in these critical dynamics models.[6] In this paper I study the phenomenon of collective oscillations in a simple model with short-range interactions.[7] As results, I find the following.

(i) Any initial probability density evolves toward a stationary or time-periodic distribution as time goes to infinite (in perturbation theory).

(ii) There is a Goldstone mode associated with the temporal symmetry breaking which gives rise to long-tail behavior beyond the critical point.

(iii) Near the onset of the oscillations, a renormalization-group (RG) analysis is performed. The critical exponents and the effective equation for the mean value are found to one loop in the $4-d$ expansion. This analysis indeed shows that the transition to the "temporal dissipative structure" in my model is analogous to a second-order equilibrium phase transition. Hopefully, this work may pave the way for a nonequilibrium statistical mechanics of such time-dependent phenomena.

I study here a model of self-synchronizing oscillators governed by the following simple equation for the two-component oscillator field $\mathbf{u}(\mathbf{x},t)$ ($\mathbf{x}$ is d dimensional):

$$\partial\mathbf{u}/\partial t = D\Delta\mathbf{u} + (\alpha - \lambda_1\mathbf{u}^2)\mathbf{u} + (\omega - \lambda_2\mathbf{u}^2)\mathbf{J}\mathbf{u} + T^{1/2}\mathbf{w}(\mathbf{x},t), \quad \mathbf{u} = (u_1, u_2), \tag{1a}$$

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1b}$$

In (1a), $\mathbf{w}(\mathbf{x},t)$ is a Gaussian white noise with zero mean and correlation

$$\langle \mathbf{w}(\mathbf{x},t)\mathbf{w}(\mathbf{x}',t')\rangle = \mathbf{1}\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')$$

($\mathbf{1}$ is the 2×2 identity matrix). $\mathbf{u}^2$ stands for the scalar product $\mathbf{u}\cdot\mathbf{u} = u_1^2 + u_2^2$. Consider the deterministic version of

(1) with $T=D=0$. We have then the simplest equation for a single nonlinear oscillator undergoing a Hopf bifurcation[8] toward a time-periodic solution: For $\alpha<0$, $\mathbf{u}=\mathbf{0}$ is a stable solution of this equation. For $\alpha>0$, the homogeneous solution

$$\mathbf{u}=(\alpha/\lambda_1)^{1/2}(\cos\omega't,\sin\omega't),\quad \omega'=(\omega-\alpha\lambda_2/\lambda_1), \tag{2}$$

is stable except for a constant phase shift. Equation (1) refers to a collection of infinitely many such oscillators interacting via a short-range diffusive term and in contact with a thermal bath at temperature T.

The mean-field version of (1) was analyzed in Ref. 3. In this version, the space variable is discretized: We use $\mathbf{u}_j(t)$, $j\in\mathbb{Z}^d$, instead of $\mathbf{u}(\mathbf{x},t)$, $\mathbf{x}\in\mathbb{R}^d$. The difference between $\mathbf{u}_j$ and the arithmetic mean of all $\mathbf{u}_j$'s is then substituted for the Laplacian term $D\Delta\mathbf{u}$. As in the case of ordinary phase transitions, the mean-field version is free from the pathologies that plague the model with short-range interactions (see below). Because of this good behavior, the probability density associated with the mean-field system may be built by standard techniques of bifurcation theory.[9]

To study Eq. (1) one can use the same WKB (Wentzel-Kramers-Brillouin) method as in Ref. 4. This yields an approximate equation for the mean value of the solution of the stochastic equation valid in the limit $T\to 0$: To leading order, the mean obeys Eq. (1) with $T=0$. Once the mean is known, equations for the correlation and response functions are found by the following scheme: First add an external field to the right side of (1); then solve (1) by iteration in powers of the nonlinearities. The leading-order approximation to the response function is found by our averaging the result and computing the term linear in the external field. The leading-order approximation to the correlation function is found by our computing

$$\mathbf{C}(\mathbf{x}-\mathbf{x}',tt')\equiv\langle\mathbf{u}(\mathbf{x},t)\mathbf{u}(\mathbf{x}',t')\rangle-\langle\mathbf{u}(\mathbf{x},t)\rangle\langle\mathbf{u}(\mathbf{x}',t')\rangle$$

from the iterative solution of (1) at zero external field.[10] Let me briefly list the main results thus obtained.

I. The stable solutions for (1) with $T=0$ are $\langle\mathbf{u}\rangle=\mathbf{0}$ if $\alpha<0$ and Eq. (2) if $\alpha>0$. In the latter case, it is possible to find linearly stable periodic traveling-wave solutions in addition to (2).[11] These waves are long-wavelength excitations of (2) to which I will not give further consideration here. These results *suggest* a phase transition from a quiescent to an oscillatory phase characterized by a nonzero order parameter (2).

II. There is an interesting phenomenon associated with the temporal symmetry breaking at $\alpha=0$. It can be illustrated by our writing the response and correlation functions below and above the critical point. These functions are given below for $\langle\mathbf{u}\rangle$ equal to the stable solution of (1) at $T=0$. Let me take $\omega=1$, $\lambda_1=1$, and $\lambda_2=0$ for simplicity (the general case yields more complicated equations but adds no new features). Then

$$\langle\mathbf{u}(t)\rangle=\alpha^{1/2}\theta(\alpha)\mathbf{e}(t),\quad \mathbf{e}(t)\equiv(\cos t,\sin t)^{\dagger}. \tag{3}$$

For $\alpha<0$, the response function is

$$\mathbf{R}(\mathbf{x}-\mathbf{x}',t,t')=\theta(t-t')[\pi D(t-t')]^{-d/2}\exp\{\alpha(t-t')-(\mathbf{x}-\mathbf{x}')^2/[4D(t-t')]\}\mathbf{E}(t-t'), \tag{4}$$

$$\mathbf{E}(t)\equiv(\mathbf{e}(t),d\mathbf{e}(t)/dt). \tag{5}$$

$\mathbf{R}$ and $\mathbf{E}$ are 2×2 matrix functions. In (3) the dagger means matrix transposition. $\theta(\alpha)$ is the Heaviside step function equal to 1 (0) if $\alpha>0$ ($\alpha<0$). Notice that we have chosen a convenient origin of time in (3). The correlation function is

$$\mathbf{C}(\mathbf{x}-\mathbf{x}',t,t')=\tfrac{1}{2}T\mathbf{E}(t-t')\int d^3k\exp\{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-(D\mathbf{k}^2-\alpha)\,|t-t'|\}(D\mathbf{k}^2-\alpha)^{-1}. \tag{6}$$

A factor $(2\pi)^{-d}$ is included in the definition of d^3k. On the other hand, for $\alpha>0$ we have

$$\mathbf{R}(\mathbf{x}-\mathbf{x}',t,t')=\theta(t-t')[\pi D(t-t')]^{-d/2}\exp\{-(\mathbf{x}-\mathbf{x}')^2/[4D(t-t')]\}\{\mathbf{A}(t,t')+\exp[-2\alpha(t-t')]\mathbf{B}(t,t')\}, \tag{7}$$

$$\mathbf{C}(\mathbf{x}-\mathbf{x}',t,t')=\mathbf{A}(t,t')g(\mathbf{x}-\mathbf{x}',t,t';0)+\mathbf{B}(t,t')g(\mathbf{x}-\mathbf{x}',t,t';\alpha), \tag{8}$$

$$g(\mathbf{x},t,t';\alpha)=\tfrac{1}{2}T\int d^3k\exp\{-(D\mathbf{k}^2+2\alpha)\,|t-t'|\}\exp[-i\mathbf{k}\cdot\mathbf{x}]/(D\mathbf{k}^2+2\alpha). \tag{9}$$

In (7) and (8) the matrices $\mathbf{A}(t,t')$ and $\mathbf{B}(t,t')$ are defined by

$$\mathbf{A}(t,t')=\begin{pmatrix}\sin t\sin t' & -\sin t\cos t'\\ -\sin t'\cos t & \cos t\cos t'\end{pmatrix}, \tag{10}$$

$$\mathbf{B}(t,t')=\begin{pmatrix}\cos t\cos t' & \sin t'\cos t\\ \sin t\cos t' & \sin t\sin t'\end{pmatrix}. \tag{11}$$

Response and correlation functions oscillate with time below and above the critical point $\alpha=0$. However, below the critical point, the envelope of the maxima of the correlation function decays exponentially to zero as time increases. At and above the critical point, this envelope decays algebraically, as $|t-t'|^{-d/2}$, where d is the space dimensionality.

This latter feature of the self-sustained oscillations is peculiar to the short-range interaction in the present problem. In the mean-field version of this model, the correlation does not decay above the critical line.[9] The algebraic decay [terms proportional to $\mathbf{A}(t,t')$ in (7) and (8)] is the equivalent of the Goldstone boson in field theory: It means that broken-symmetry solutions differing in a constant phase shift from (3) represent equivalent states. The term proportional to $\mathbf{B}(t,t')$ in (7) and (8) indicates the stability of the state given by the order parameter (3): It decays exponentially at a rate given by the Floquet exponent[8] of (3) [considered as a solution of the deterministic equation (1) with $T=D=0$]. It is only at the critical point $\alpha=0$ that this term decays algebraically.

III. The explicit form of mean value, correlation functions, and higher moments of $\mathbf{u}$ depend on the probability distribution at some initial time. At leading order of approximation, any initial condition tends exponentially to the above expressions as time goes to infinity (except for a constant phase shift). For example, had I chosen a zero correlation at $t=0$, I would have to write

$$\{\exp[-(D\mathbf{k}^2+2\alpha)\,|t-t'|\,]-\exp[-(D\mathbf{k}^2+2\alpha)(t+t')]\}$$

instead of $\exp\{-(D\mathbf{k}^2+2\alpha)\,|t-t'|\}$ in (6) and (8). In the limit $t, t' \to \infty$, with $|t-t'|$ fixed, we get back (6) and (8), which correspond to having zero correlation at $t=-\infty$. This argument can be extended to higher-order terms in the perturbation expansion of all moments of $\mathbf{u}$. If I am only interested in long-time results, I can therefore restrict my analyses to the case of a probability density of $\delta(u-\mathbf{U}(\mathbf{x}))$ at time $-\infty$.

One typically hopes this Landau-Ginzburg description to hold qualitatively at the critical region, near $\alpha=0$. Near the critical point, I have performed a renormalization-group (RG) calculation. First of all, let me define the following complex variables and parameters:

$$u_1+iu_2=e^{i\omega t}u,\quad \lambda_1+i\lambda_2=\lambda,\quad w_1+iw_2=e^{i\omega t}w. \tag{12}$$

(12) transforms (1) into a complex Langevin equation and sets the oscillation frequency equal to zero at the critical point $\alpha=0$. The resulting equation can be analyzed by the path-integral version of the Martin-Siggia-Rose (MSR) formalism.[12] I have proved[13] that the theory thus obtained can be renormalized to all orders in perturbation theory by definition of the following renormalized fields and parameters (zero subscripts indicate nonrenormalized quantities):

$$\begin{aligned}
&\alpha_0/D_0\equiv\alpha_0(1+i\gamma_0)=\mu^2 Z_\alpha\alpha(1+iZ_\gamma\gamma),\\
&D_0\equiv\delta_0(1+iv_0)=Z_\delta\delta(1+ivZ_v),\\
&\lambda_0/D_0\equiv\lambda_{10}+i\lambda_{20}=\mu^{\epsilon}(Z_1+iZ_2)(\lambda_1+i\lambda_2),\\
&T_0\equiv 2\delta_0,\quad v_0=\mu^{3-\epsilon/2}Z_v^{1/2}v,\\
&u_0=\mu^{1-\epsilon/2}Z_u^{1/2}u.
\end{aligned} \tag{13}$$

Except for Z_u and Z_v, all the Z's are real. They are a power series in λ_1, λ_2, and v whose coefficients depend on $\epsilon\equiv 4-d$ and γ. Notice that renormalization to all orders requires the introduction of bare and renormalized imaginary parts of the control parameter α and of the diffusion coefficient D. $\alpha\gamma$ is interpreted as a correction to the frequency of the collective oscillations, while v does not have an easy interpretation.[14] $\mu>0$ is a parameter with dimensions of 1/length.

I have calculated the RG equations and the corresponding critical exponents following the standard procedures.[12] To one-loop order and $d<4$, the RG equations have the unique infrared stable fixed point $\lambda_1=3\epsilon/10$, $\lambda_2=0$, and $\gamma=0$. This fixed point is the same one as in the critical dynamics of the two-component ϕ^4 model.[15] Correspondingly, I find that the (dimensionful) order parameter, $M\equiv\mu^{1-\epsilon/2}|\langle u(\mathbf{x},t)\rangle|$, the relaxation time τ, the correlation length ξ, and the correction to the frequency of the collective oscillations $\alpha\gamma$ scale as

$$\begin{aligned}
&M\sim\alpha^{1/2-2\epsilon/5},\quad \tau\sim\alpha^{-1-\epsilon/5},\\
&\xi\sim\alpha^{-1/2-\epsilon/10},\quad \alpha\gamma\sim\alpha^{1+\epsilon/5}\quad(\alpha\to 0+).
\end{aligned} \tag{14}$$

It is only for $d>4$ that we find the same exponents as in the deterministic case, corresponding to $\epsilon=0$ in (14).[15] By means of the saddle-point method, a universal effective equation for $\langle u(\mathbf{x},t)\rangle$ can be derived.[10,13,16] This equation has the linearly stable time-periodic solution $M\exp[-i\alpha\gamma t]$. The effective equation is asymptotically valid as $\alpha\to 0$, and it describes the slow space-time evolution [the scaling of time and space is given by that of τ and ξ in (14)] of the initial $\langle u\rangle$ toward the stable oscillatory state. The derivation of the effective equation,[13,16] together with $\lambda_2=0$ at the RG fixed point, implies the following: Above the critical point, the stable oscillatory state is a time-dependent rotation of the equilibrium state of the two-component ϕ^4 model. I shall discuss elsewhere the derivation and properties of the effective equation[13] which is the analog of the equation of state for equilibrium phase transitions.[15]

In conclusion, I have analyzed a simple model having a second-order nonequilibrium phase transition from a quiescent to a self-sustained oscillatory state. At the level of the tree approximation,[15] I obtain the most important qualitative features of the model, with the wrong critical exponents.[15] Near the critical point, thermal fluctuations have to be considered via RG calculations. I find that the fixed point of the RG is the same as in the critical dynamics of the two-component ϕ^4 model. Disturbances from the oscillatory state evolve according to a universal effective equation for the order parameter. I thus find that the oscillatory state is a rotating equilibrium state with the consequences of stability, generalized rigidity,[17] etc. To my knowledge, this is the first time a RG analysis of a "temporal dissipative structure"[18] has been completed.

Let me add a final remark. Repeating my calculations for the sine-Gordon nonlinearity in the presence of a constant external field, I also find a nonequilibrium phase transition to a time-periodic probability density.[19] This shows a definite limitation to the use of the stochastic quantization procedure.[20] Even though there exists an equilibrium state for all external fields, above a certain critical field the probability tends to a periodic function of time as $t \rightarrow \infty$. Thus we cannot interchange the thermodynamic and long-time limits in this case, and the method of stochastic quantization is not applicable.

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[1]Y. Yamaguchi, K. Komentani, and H. Shimizu, J. Stat. Phys. **26**, 719 (1981); Y. Kuramoto, Physica (Amsterdam) **106A**, 128 (1981); R. C. Desai and R. Zwanzig, J. Stat. Phys. **19**, 1 (1978); D. A. Dawson, J. Stat. Phys. **31**, 29 (1983).

[2]M. Scheutzow, Probab. Th. Rel. Fields **72**, 425 (1986).

[3]M. Shiino, Phys. Lett. **111A**, 396 (1985).

[4]L. L. Bonilla, Phys. Rev. B **35**, 3637 (1987).

[5]W. Horsthemke and R. Lefever, *Noise-Induced Transitions* (Springer-Verlag, Berlin, 1984). See, however, H. Lemarchand and G. Nicolis, Physica (Amsterdam) **82A**, 521 (1976) where the thermodynamic limit of a multivariate master equation is analyzed.

[6]P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977); S.-K. Ma and G. F. Mazenko, Phys. Rev. B **11**, 4077 (1975); C. De Dominicis and L. Peliti, Phys. Rev. B **18**, 353 (1978).

[7]Of course we can consider other stable time-dependent states (e.g., chaotic), not just time-periodic ones. Short-range models having these nonequilibrium states can be built as indicated in Ref. 4: Add a diffusion term and a white noise forcing term to the corresponding deterministic equation.

[8]In fact, close to the bifurcation point, any problem having a Hopf bifurcation can be reduced to the normal form (1) with $D = T = 0$ by projection techniques. See V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations* (Springer-Verlag, New York, 1983). Also G. Iooss and D. D. Joseph, *Elementary Stability and Bifurcation Theory* (Springer-Verlag, New York, 1980), Chaps. 7 and 8. The projection technique has been recently extended to stochastic equations by G. Schöner and H. Haken, Z. Phys. B **68**, 89 (1987). The result for the case of a general Hopf bifurcation seems to be Eq. (1) plus terms with multiplicative noise. Thus (1) would be a simplification of a general normal form for stochastic reaction-diffusion equations having a Hopf bifurcation.

[9]L. L. Bonilla, J. Stat. Phys. **46**, 659 (1987); L. L. Bonilla, J. M. Casado, and M. Morillo, J. Stat. Phys. **48**, 571 (1987), and **50**, 849(E) (1988).

[10]See Ma and Mazenko, Ref. 6, and also L. L. Bonilla, in *Random Media,* edited by G. Papanicolaou, IMA Volumes in Mathematics and Its Applications Vol. 7 (Springer-Verlag, New York, 1987), p. 15. As indicated in this paper, the iteration scheme alluded to in the text is equivalent to the Martin-Siggia-Rose (MSR) formulation which I use later. The results presented in Eqs. (3)–(11) correspond to the tree approximation in the MSR formalism.

[11]N. Kopell and L. N. Howard, Stud. Appl. Math. **52**, 291 (1973).

[12]See, for example, De Dominicis and Peliti, Ref. 6.

[13]L. L. Bonilla, unpublished. The RG calculations are similar to those by De Dominicis and Peliti in Ref. 6.

[14]For deterministic reaction-diffusion equations having a Hopf bifurcation, such an imaginary part appears in the normal form (1) as a result of the projection technique. See, e.g., Y. Kuramoto, in *Synergetics—A Workshop,* edited by H. Haken (Springer-Verlag, Berlin, 1977).

[15]D. J. Amit, *Field Theory, the Renormalization Group and Critical Phenomena* (World Scientific, Singapore, 1984), 2nd ed., p. 236. Notice that for $\lambda_2 = 0$ in (1), the state of collective oscillations corresponds to a time-dependent rotation of the equilibrium state of the two-component ϕ^4 model. The exponents in (14) are the same as in the ϕ^4 model except for the correction to the collective frequency which has no analog in the equilibrium case.

[16]To handle the MSR theory above the critical point, where $\langle u(\mathbf{x},t)\rangle$ is oscillatory, I performed a convenient rotation of the fields u, v in the MSR path integral which effectively set $\langle u\rangle$ equal to a slowly varying function of $\mathbf{x}$ and t. Then $\langle u\rangle$ can be treated approximately as a constant in the resulting MSR theory. I will give the details elsewhere.

[17]P. W. Anderson, *Basic Notions of Condensed-Matter Physics* (Benjamin, Menlo Park, CA, 1984), Chap. 2, p. 262.

[18]G. Nicolis and I. Progogine, *Self-Organization in Nonequilibrium Systems* (Wiley, New York, 1977).

[19]The correspondence equation

$$\partial\phi/\partial t = D\Delta\phi + E - h\sin\phi + T^{1/2}w(\mathbf{x},t),$$

$$\langle w\rangle = 0,\quad \langle w(\mathbf{x},t)w(\mathbf{x}',t')\rangle = \delta(\mathbf{x}-\mathbf{x}')\delta(t-t'),$$

was analyzed (in the mean-field approximation) in Ref. 4, in relation with the depinning of charge-density waves. For $E > h$, the stable solution of the deterministic homogeneous equation is time periodic. A nonequilibrium phase transition from a stationary to a time-periodic probability density associated to $\phi(\mathbf{x},t)$ occurs at $E = h$ (to leading order in T) (Ref. 4). The calculations in the present paper can be extended to cover this short-range model. I will do this elsewhere.

[20]G. Parisi and Y. Wu, Sci. Sin. **24**, 483 (1981); E. Floratos and J. Ilipoulos, Nucl. Phys. **B214**, 392 (1983); A. Gonzalez-Arroyo, in *Applications of Field Theory to Statistical Mechanics,* edited by L. Garrido, Lecture Notes in Physics Vol. 216 (Springer-Verlag, New York, 1985), p. 171.