

Two-Dimensional Guiding-Center Transport of a Pure Electron Plasma

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Cross-field particle transport is discussed for the case of a magnetically confined pure-electron-plasma column. The analysis is based on a 2D guiding-center model and yields a flux which scales with magnetic field strength like $1/B$. The magnitude of the flux exceeds that predicted previously provided that two conditions are met. The axial bounce frequency for a typical electron must exceed the $\mathbf{E} \times \mathbf{B}$ drift rotation frequency, and this rotation frequency must be a nonmonotonic function of radius.

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Recently, a guiding-center model was introduced to describe like-particle transport in a magnetically confined pure-electron-plasma column,¹ and this paper provides a further development of the model. The cross-field velocity of an electron is given by $\mathbf{v}_E = c(\mathbf{E} \times \mathbf{B})/B^2$, where \mathbf{B} is the uniform axial magnetic field and \mathbf{E} is the electric field produced by the other electrons. The electric field consists of a mean-field component $\langle E_r(r) \rangle$ in the radial direction (because the plasma is nonneutral) and a fluctuating component. The mean field produces a zeroth-order drift rotation, with frequency $\omega_E(r) = -c\langle E_r(r) \rangle/rB$, and the fluctuating field produces the radial flux to be calculated. This model provides a good approximation² when the characteristic cyclotron radius and cyclotron period are small compared with the length and time scales of interest in the field \mathbf{E} , and that is the case here.

To understand the distinction between the present work and the previous work on this model,¹ we first note that the fluctuating field produced by electron 2 is effective in producing transport of electron 1 only if electrons 1 and 2 interact resonantly. They must satisfy a resonance condition of the form $kv_1 + l\omega_E(r_1) = kv_2 + l\omega_E(r_2)$, where v_1 and v_2 are the velocities of electrons 1 and 2 along the magnetic field, and $\omega_E(r_1)$ and $\omega_E(r_2)$ are the rotation frequencies. The interaction is effected through a Fourier component characterized by axial wave number k and azimuthal mode number l . Here, we have used the azimuthal symmetry of the system and have imposed periodic boundary conditions along the field (i.e., $k = 2\pi n/L$).

The previous work focuses attention on the Debye-shielded portion of the interaction field; that is $k \approx 1/\lambda_D$, where λ_D is the Debye length. Since $\bar{v}/\lambda_D = \omega_p$ is much larger than ω_E in the typical operating regime of pure-electron-plasma experiments (where \bar{v} is the thermal velocity and ω_p is the plasma frequency), the resonance condition reduces to $kv_1 = kv_2$. In contrast, here we focus attention on Fourier components for which $k\bar{v} \ll \omega_E$, so that the resonance condition reduces to $l\omega_E(r_1) = l\omega_E(r_2)$.

This latter resonance condition can be satisfied for r_1

$\neq r_2$ only if the rotation frequency $\omega_E(r)$ is nonmonotonic; of course, the self-interaction at $r_1 = r_2$ does not produce transport. For a current series of experiments,³ density profiles corresponding to nonmonotonic $\omega_E(r)$ are often used, and it is these experiments which motivate the present theory. An order-of-magnitude comparison shows that the present flux is larger than that calculated previously, provided that $\bar{v}/L > \omega_E$, and this is the operating regime of the experiments. Incidentally, the only Fourier components for which $k\bar{v} \ll \omega_E$, $\bar{v}/L > \omega_E$, and $k = 2\pi n/L$ are those for $k = 0$, and so we consider only these components.

The flux calculated here scales differently with magnetic field strength than does that calculated previously. To understand this difference, simply note that the effective interaction time for electrons 1 and 2 is set by the mismatch in the resonance. For the previous treatment, the interaction time is set by differential streaming along the field lines {i.e., $[k(v_1 - v_2)]^{-1}$ }, which is independent of field strength, whereas here the interaction time is set by differential rotation {i.e., $[l\omega_E(r_1) - l\omega_E(r_2)]^{-1}$ }, which scales like B . The flux obtained previously scales like B^{-2} , whereas that obtained here scales like B^{-1} .

This latter scaling is in agreement with observation for experiments³ which satisfy the two conditions that $\omega_E(r)$ is nonmonotonic and $\bar{v}/L > \omega_E$. Of course, the experiments have been designed to ensure that like-particle transport dominates over transport due to other effects such as field errors.⁴

Our focusing attention on the $k = 0$ Fourier components is equivalent to considering the 2D dynamics of charged rods. There has been much theoretical work on 2D transport for the case of a neutral plasma,⁵ and it is instructive to understand the relationship of that work to the present work. The basic difference is that in a neutral plasma the rotation frequency is taken to be zero, and so the resonance condition $\omega_E(r_1) = \omega_E(r_2)$ is trivially satisfied everywhere. This multiplicity of resonant interactions makes the problem much harder than the present problem. In technical terms, the rotation frequency $\omega_E(r)$ provides a linear term in the propagator,

and when this term vanishes (or is constant) over a range of r values, the problem becomes fully nonlinear.

Another problem that is closely related to the present work is the recent quasilinear analysis of the diocotron instability.⁶ A plasma column with a nonmonotonic $\omega_E(r)$ (corresponding to a nonmonotonic density profile) can be unstable to diocotron modes. When the modes grow to large amplitude, they react back on the density distribution, producing rapid transport in a stabilizing direction. Here, however, we consider the simpler case where $\omega_E(r)$ is nonmonotonic, but the plasma is still stable (in accord with the experiments), and the transport is produced by fluctuations in the field, rather than by unstable modes.

For certain simple profiles, (e.g., square profiles), one can show that the diocotron modes are stable if the nonmonotonicity is sufficiently mild. However, we do not justify the assumption of stability on the basis of a theoretical argument, which to be complete would require the analysis of arbitrary smooth profiles and finite length effects. Rather, we rely on the experimental observation that the plasma is stable for certain nonmonotonic profiles, and we intend our theory to apply only to such profiles.

Proceeding now to the analysis itself, we define the Klimontovich density distribution⁷ as

$$\mathcal{N}(r, \theta, t) = \sum_{j=1}^N \frac{1}{L} \frac{\delta(r - r_j(t)) \delta(\theta - \theta_j(t))}{r}, \quad (1)$$

where $[r_j(t), \theta_j(t)]$ specifies the transverse position of the j th electron. Here, each electron has been averaged over z to form a rod of length L . Note that $r_j(t)$ and $\theta_j(t)$ are implicit functions of the initial positions $\{r_1(0), \theta_1(0), \dots, r_N(0), \theta_N(0)\}$. For brevity, we will denote the initial positions by $\{r_1, \theta_1, \dots, r_N, \theta_N\}$. The

distribution evolves according to the equation

$$[\partial/\partial t + (c/B)\hat{z} \times \nabla \Phi \cdot \nabla] \mathcal{N} = 0, \quad (2)$$

where $\Phi(r, \theta)$ is the self-consistent potential (i.e., $\nabla^2 \Phi = 4\pi e \mathcal{N}$). As a boundary condition, we take $\Phi(r, \theta)$ to be constant on a cylindrical conducting wall at radius $r = R$.

Following the usual Klimontovich procedure,⁷ we define the average density $n(r) = \langle \mathcal{N} \rangle$, where the angular brackets signify an ensemble average over uncorrelated initial positions. The correlations are allowed to develop through the dynamics. The density fluctuation is given by $\delta n(r, \theta, t) = \mathcal{N}(r, \theta, t) - n(r)$. Likewise, we define an average potential (mean field) $\phi(r) = \langle \Phi \rangle$ and a potential fluctuation $\delta \phi(r, \theta, t) = \Phi(r, \theta, t) - \phi(r)$. Of course, $\phi(r)$ and $\delta \phi(r, \theta, t)$ are related individually to $n(r)$ and $\delta n(r, \theta, t)$ through Poisson's equation.

By taking the average of Eq. (2), we obtain a simple expression for the radial flux,

$$\Gamma_r(r) = - \left(\frac{c}{Br} \right) \left\langle \delta n \frac{\partial \delta \phi}{\partial \theta} \right\rangle, \quad (3)$$

and by subtracting the average of Eq. (2) from Eq. (2) itself, we obtain an equation for the density fluctuation δn . To lowest order, we may drop terms in this equation that are quadratic in fluctuating quantities; the result is

$$\left(\frac{\partial}{\partial t} + \omega_E(r) \frac{\partial}{\partial \theta} \right) \delta n - \frac{c}{Br} \frac{\partial \delta \phi}{\partial \theta} \frac{\partial n}{\partial r} = 0, \quad (4)$$

where $\omega_E(r) = (c/Br) \partial \phi / \partial r$. Incidentally, for the 2D dynamics of charged rods in a neutral plasma,⁵ $\omega_E(r)$ and $\partial n / \partial r$ are both taken to be zero; so the dynamics is dominated by the nonlinear terms that may be neglected here.

By Fourier analysis in θ and Laplace transformation in t , Eq. (4) reduces, for $l \neq 0$, to

$$[p + i l \omega_E(r)] \delta n(r, l, p) - \frac{c i l}{Br} \frac{\partial n}{\partial r} \delta \phi(r, l, p) = \sum_{j=1}^N \frac{1}{L} \frac{\delta(r - r_j) e^{-i l \theta_j}}{r_j 2\pi}, \quad (5)$$

where the right-hand side is an explicit expression for $\delta n(r, l, t=0)$. This equation must be solved simultaneously with Poisson's equation, which also relates $\delta \phi(r, l, p)$ and $\delta n(r, l, p)$. One can easily check that the solution for $\delta \phi(r, l, p)$ is given by

$$\delta \phi(r, l, p) = \sum_{j=1}^N \frac{4\pi e}{2\pi L} \frac{e^{-i l \theta_j}}{p + i l \omega_E(r_j)} \delta \psi(r, r_j, l, p), \quad (6)$$

where

$$\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{l^2}{r^2} - \frac{4\pi e c i l}{Br} \frac{\partial n}{\partial r} \frac{1}{p + i l \omega_E(r)} \right) \delta \psi(r, r_j, l, p) = \frac{\delta(r - r_j)}{r}. \quad (7)$$

The solution for $\delta n(r, l, p)$ is now given trivially by Eq. (5).

The substitution of these solutions into expression (3) for the flux and the carrying out of the average over uncorrelat-

ed initial positions yields the result

$$\Gamma_r(r) = -\frac{c}{Br} \left(\frac{4\pi e}{2\pi L} \right)^2 \sum_l \int \frac{dp dp'}{(2\pi i)^2} e^{(p+p')t} \int_0^R 2\pi L r_j dr_j n(r_j) \frac{(-il)\delta\psi(r, r_j, -l, p')}{[p' - il\omega_E(r_j)][p + il\omega_E(r_j)]} \times \left(\frac{cil}{Br} \frac{\partial n}{\partial r} \frac{\delta\psi(r, r_j, l, p)}{p + il\omega_E(r)} + \frac{\delta(r - r_j)}{4\pi e r_j} \right), \quad (8)$$

where the p and p' integrals are to be carried out along a path to the right of any singularities.

At this point, we make the Bogoliubov *Ansatz*⁸ and assume that the fluctuations relax to their asymptotic form on a time scale which is short compared with the transport time. To be specific, we evaluate the p and p' integrals using the Cauchy residue method retaining only the poles along the imaginary axis [e.g., $p + il\omega_E(r) = 0$]. The poles and branch cuts associated with $\delta\psi$ are assumed to give damped contributions, and t is chosen to be much larger than the damping time but still small compared with the transport time. The result of this evaluation is

$$\Gamma_r(r) = -\frac{c}{Br} \frac{(4\pi e)^2}{2\pi L} \sum_l \int_0^R r_j dr_j n(r_j) \left\{ \frac{l^2 c}{Br} \frac{\partial n}{\partial r} \left| \delta\psi[r, r_j, l, -il\omega_E(r)] \right|^2 \pi \delta(l\omega_E(r) - l\omega_E(r_j)) + \frac{l\delta(r - r_j)}{4\pi e r_j} \text{Im} \delta\psi^*[r, r_j, l, -il\omega_E(r)] \right\}, \quad (9)$$

where use has been made of the reality condition $\delta\psi^*[r, r_j, l, -il\omega_E(r)] = \delta\psi[r, r_j, -l, il\omega_E(r)]$.

It is convenient to rewrite the last term in the braces. To this end, set $p = il\omega_E(r_j) + \epsilon$ in Eq. (7) and multiply both sides of the equation by $\delta\psi^*[r, r_j, l, -il\omega_E(r_j)]$. Integration of the resulting equation over $r dr$ and the taking of the imaginary part of both sides yields the relation

$$\text{Im} \delta\psi^*[r, r_j, l, -il\omega_E(r_j)] = -(4\pi e c l / B) \int_0^R dr (\partial n / \partial r) \left| \delta\psi[r, r_j, l, -il\omega_E(r)] \right|^2 \pi \delta(l\omega_E(r) - l\omega_E(r_j)). \quad (10)$$

Replacement of the dummy variable r by r' in Eq. (10) and substitution of this relation into Eq. (9) then yields the desired result for the flux,

$$\Gamma_r(r) = \frac{(4\pi e)^2 c^2}{2LB^2 r} \sum_l |l| \int_0^R r' dr' \left| \delta\psi[r, r', l, -il\omega_E(r)] \right|^2 \delta(\omega_E(r) - \omega_E(r')) \left(\frac{1}{r'} \frac{\partial}{\partial r'} - \frac{1}{r} \frac{\partial}{\partial r} \right) n(r) n(r'). \quad (11)$$

As one expects for transport produced by resonant interactions, the flux is nonzero unless $\omega_E(r) = \omega_E(r')$ for $r \neq r'$, that is, unless $\omega_E(r)$ is nonmonotonic. Also, the expression for the flux has the expected formal properties. To see that the transport conserves the total canonical angular momentum [i.e., $P_\theta = \int r dr (eB/2c) r^2 \times n(r, t)$], as one expects for a plasma confined in a cylindrically symmetrical geometry, note that the integral $\int r dr \Gamma_r(r) r$ vanishes by antisymmetry under the interchange of r' and r . To see that the transport conserves the total electrostatic energy, as one expects for $\mathbf{E} \times \mathbf{B}$ dynamics, note that the integral $\int r dr E_r \Gamma_r(r)$ vanishes by antisymmetry under interchange of r' and r . Here, one must use the relation $E_r(r) = -(B/c)r\omega_E(r)$. Also, one can easily check that the transport increases an entropy function, that is, $dS/dt \geq 0$, where

$$S = - \int r dr n(r, t) \ln[n(r, t)].$$

In the experiments,³ $\omega_E(r)$ has a single peak, and so for a range of r values there are two corresponding points such that $\omega_E(r) = \omega_E(r')$ with $r \neq r'$, and at such points the flux need not vanish. To estimate the order of magnitude of the flux at one of these points, we replace the sum $\sum_l |l| |\delta\psi|^2$ by unity [this is an underestimate if

the frequency $l\omega_E(r)$ is near the frequency of a normal mode] and retain only one of the two derivative terms. The result is

$$\Gamma \sim 2\pi \left[\frac{\omega_E r n}{N} \right] \frac{\partial \ln n(r) / \partial r}{\partial \ln \omega_E(r) / \partial r}, \quad (12)$$

where the first bracket is the zeroth-order azimuthal flux divided by the total number of electrons ($N \approx n\pi^2 L$) and where we have set $\omega_E = (2\pi n e c) / B$.

The Klimontovich analysis presented here can be generalized to include interactions for $k \neq 0$ (as will be shown in a future publication), and by focusing attention on the contribution from wave numbers in the range $k \sim 1/\lambda_D$ one recovers the result obtained previously by use of the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy.¹ The ratio of the flux calculated here [Eq. (12)] to that calculated previously is approximately $(\bar{v}/\omega_E L) [10^2 / \ln(\bar{v}/\Delta v)]$, where the factor $\ln(\bar{v}/\Delta v)$ was defined previously¹ and may be taken to be of order 10. The flux calculated here dominates when the axial bounce frequency ($\bar{v}L$) exceeds the rotation frequency ω_E . Of course, this assumes that $\omega_E(r)$ is a nonmono-

tonic function of r .

Finally, we note that the nonlinear terms neglected in Eq. (4) would broaden the resonance condition $\omega_E(r) = \omega_E(r')$. One expects the broadening, ν , to satisfy the relation $\nu^2 = l^2 (\partial \omega_E / \partial r)^2 (\Delta r)^2$, where $(\Delta r)^2 = D/\nu$ and D is the test-particle diffusion coefficient. To estimate D we set $\Gamma \sim D \partial n / \partial r$. One sees that the resonant broadening is small (i.e., $\nu \ll r l |\partial \omega_E / \partial r|$) provided that $\tau l |\partial \omega_E / \partial r| \gg 1$, where $\tau = nr/\Gamma$ is the transport time. In the experiments, this is the case over most of the plasma radius.

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