High-Energy Symmetries of String Theory

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By means of a recent analysis of the high-energy limit of string scattering, linear relations between string-scattering amplitudes are derived. These are shown to hold order by order in perturbation theory. If one assumes that they hold for the full theory, they suggest the existence of an enormous string-broken symmetry which is restored at high energies. Some speculations as to the nature of this symmetry are presented.

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It is often the case that spontaneously broken symmetries of a physical theory are hard to recognize at low energy, but become evident in the high-energy behavior of the theory. Thus, the broken $SU(2)\otimes U(1)$ symmetry of the electroweak interactions can be seen by examination of weak scattering amplitudes at energies high enough that the W and Z_0 masses can be neglected. String theory surely possesses a very rich symmetry, as suggested by its incredible degree of uniqueness; however, this symmetry is little understood. Presumably this is because most of the string symmetry is spontaneously broken in the known ground states, leaving only the familiar gauge symmetries unbroken and manifest. Perhaps all the string states are gauge particles, but most are massive because of spontaneous symmetry breaking. Perhaps at very high energies, so high that the Planck mass (which in string theory is proportional to the string tension $T=1/\pi\alpha'$) can be neglected, the full symmetry of string theory is restored. In this limit, of course, all particles have vanishing mass, $P^2=(1/\alpha')\times\text{integer}\rightarrow 0$. An even wilder speculation is that the Planck mass itself arises dynamically from a more symmetric scale-invariant phase of string theory. Perhaps we can discover this symmetry (if it exists) by studying string theory in the high-energy (or $\alpha'\rightarrow\infty$) limit. In particular, if at high energies a larger symmetry is restored there should then exist linear relations between the scattering amplitudes that should be valid order by order in perturbation theory. If so these might be discoverable by our analyzing the high-energy behavior of the theory perturbatively.

This limit of string theory, which can be thought of as either the high-energy limit or as the limit where $M^2_{\text{Planck}}\sim 1/\alpha'\rightarrow 0$, is of great interest. It is the opposite of the low-energy limit, $\alpha'\rightarrow 0$, where strings behave as particles and the dynamics can be represented by a local effective field theory. As stressed in the work of Gross and Mende,[1] the high-energy behavior of strings is very *stringy* and cannot be reproduced by an effective local-field theory. In ordinary general relativity this limit, which is the same as the strong-coupling limit of gravity, $G_{\text{Newton}}\propto 1/M^2_{\text{Planck}}$, is difficult to discuss since the theory breaks down in the ultraviolet. String theory does not necessarily suffer from this limitation. Finding this enlarged symmetry should help us to understand the structure of string physics at high energies. This is not just an academic issue; it is also crucial if we are to understand the Planckian-scale dynamics that determines the nature of the string ground state.

Recently, the high-energy behavior of string scattering was studied by saddle-point techniques.[1,2] It was shown that the sums over Riemann surfaces that define string perturbation theory were dominated by particular surfaces in the limit $\alpha'\rightarrow\infty$. The dominant surfaces were identified and the leading behavior of the tachyonic scattering amplitude was calculated. In this paper I shall use these results to explore for new symmetries that might be revealed at high energy. I show that *there exist an infinite number of linear relations between the scattering amplitudes of different string states that are valid order by order in perturbation theory as* $\alpha'\rightarrow\infty$. It is then not unreasonable to assume that these relations are true properties of the $\alpha'\rightarrow\infty$ limit of string theory, even though perturbation theory diverges badly. If so, they are the consequence of an infinite-parameter symmetry group which is restored as an exact symmetry at high energies. This symmetry is so powerful as to determine the scattering amplitudes of all the infinite number of string states in terms of, say, the dilaton scattering amplitudes.

First, let us recall the analysis of Ref. 1. Consider the G loop contribution to a string scattering amplitude, which is given by

$$A_G(P_i)=\int\frac{\mathcal{D}g_{\alpha\beta}}{\mathcal{N}}\mathcal{D}X^\mu\exp\left[-\frac{\alpha'}{2\pi}\int d^2\xi\sqrt{g}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu\right]\prod V_i(X_i^\mu,P_i),\tag{1}$$

where $V_i(X_i,P_i)$ is the vertex operator for particle i which carries momentum P_i,

$$V(P_i)=\int d^2\xi_i\sqrt{g}\exp[i\alpha' P_i\cdot X(\xi_i)]\mathcal{V}(X^\mu(\xi_i),P_i).$$

$\mathcal{V}(X^\mu(\xi_i),P_i)$, depends on the particular string state and its polarization, but is a polynomial in α'. The integral is over all compact surfaces of genus G. In the critical dimension it is invariant under the group of diffeomorphisms and Weyl rescalings of the metric, which is why we have divided by $\mathcal{N}$, the volume of this group. It can then be reduced to an integral over the finite-dimensional moduli space $\mathcal{M}_G$ of Riemann surfaces[3,4] of genus G parametrized by $3(G-1)$ complex moduli $\mathbf{m}$ with punctures at the positions ξ_i at which the vertex operators are located.

Here we are considering the bosonic string; however, similar results will hold for the super and heterotic string theories. I have adopted an unusual normalization for the string coordinate X (related to the more common X_n by $X=\alpha' X_n$), so as to make it clear that the $\alpha'\to\infty$ limit is, order by order in perturbation theory, equivalent to the semiclassical limit of first-quantized string theory. Thus the above integral is dominated in the limit of infinite α' by a saddle point in X^μ, $\mathbf{m}_i$, and ξ_i. This is a classical solution of the string equations of motion, $\mathbf{m}_i=\hat{\mathbf{m}}_i$, $\xi_i=\hat{\xi}_i$, and

$$X^\mu(\xi)=X^\mu_{\rm cl}(\xi)=i\textstyle\sum_i P_i^\mu G_{\hat{\mathbf{m}}}(\xi,\hat{\xi}_i)+O(1/\alpha'),\qquad(2)$$

where $G_{\hat{\mathbf{m}}}(\xi,\hat{\xi}_i)$ is the Green's function (the inverse Laplacian) on the genus-G Riemann surface with unit sources at $\hat{\xi}_i$. Consequently

$$A_G(P_i)=g^{2G+2}\Gamma_G(P_i,\hat{\mathbf{m}}_i,\hat{\xi}_i)\textstyle\prod_i\mathcal{V}_i((X^\mu_{\rm cl},\hat{\xi}_j),P_j)+O(1/\alpha'),\qquad(3)$$

where Γ_G contains an exponential factor which arises from evaluating the integrand at the saddle point (in any string theory, for a four-particle scattering amplitude whose kinematical variables are $s=(P_1+P_2)^2=(P_3+P_4)^2$, $t=(P_1+P_3)^2=(P_2+P_4)^2$, $\mu=(P_1+P_4)^2=(P_2+P_3)^2$, this was determined to be

$$\exp\{-[\alpha'/4(G+1)](s\ln s+t\ln t+u\ln u)\}$$

(Ref. 1)), as well as the measure of moduli space evaluated at the saddle-point surface and the inverse determinant of the matrix of second derivatives of the exponential at the saddle point.

The important fact to notice is that *the only factor in (3) that depends on the nature of the particles being scattered is* $\prod_i\mathcal{V}_i$! Therefore, if we can determine the saddle-point surface (in other words the $\hat{\mathbf{m}}_i$, ξ_i, and $X^\mu_{\rm cl}$, which only depend on the number of particles participating in the scattering but not on their quantum numbers), then we can immediately deduce a linear relation between any two scattering amplitudes involving the same number of particles with the same momenta (for example, the scattering of particles a_i and particles b_i),

$$A_G^{a_i}(P_i)=\frac{\prod_j\mathcal{V}_{a_j}(X^\mu_{\rm cl},P_j)}{\prod_j\mathcal{V}_{b_j}(X^\mu_{\rm cl},P_j)}A_G^{b_i}(P_i)[1+O(\alpha')].\qquad(4)$$

This is an amusing relation, but one of little use by itself since it only relates amplitudes at a given order of perturbation theory. Fortunately, for the dominant saddle points discussed in Ref. 1, the dependence of $X^\mu_{\rm cl}$ on the order of perturbation theory G is trivial. I recall that in Ref. 1 it was argued that the saddle-point surfaces that dominate elastic scattering at high energies were the Riemann surfaces of the algebraic curve $y^N=\prod_{i=1}^4(z-a_i)^{L_i}$, which describe an N-sheeted cover of the Riemann sphere (as long as the L_i are relatively prime to N) of genus $G=N-1$. The positions of the branch points, a_i, coincide with the positions of the punctures. Furthermore, the SL(2,C)-invariant cross ratio of the branch points was determined[1] to be $\lambda=(a_4-a_2)(a_1-a_3)/(a_4-a_3)(a_1-a_2)=t/s$. All of these surfaces (including those with different L_i) give rise to the same exponential factor in (3), all of them have the same Green's function, and for all of them

$$X^\mu_{\rm cl}(z)=\frac{i}{N}\sum_{i=1}^{4}P_i^\mu\ln|z-a_i|+O(1/\alpha').\qquad(5)$$

It is a remarkable fact that the saddle-point surfaces are all identical, except for the scale (which goes as $1/N$), for all the saddle points in each order of perturbation theory. Consequently, the only genus dependence present in Eq. (4) comes from the $1/N$'s in the $X^\mu_{\rm cl}$'s present in the $\mathcal{V}_i$'s, namely $\mathcal{V}(X^{\mu N}_{\rm cl},P)=\mathcal{V}(X^{\mu 1}_{\rm cl}/N,P)$. If all the $\mathcal{V}_{a_i}$'s were homogeneous functions of the X^μ's of the same degree, then (4) would be independent of G. Unfortunately, this is not the case, except for the vertex operators of the simplest string states. However, the factors of $1/N$ can be replaced by derivatives with respect to the momenta. This is because the operator $\mathcal{D}\equiv(2/\alpha' S)\sum_i P_i\cdot\partial/\partial P_i$ brings down a factor of N, when acting on $e^{-\alpha' S/4}$, where $S=s\ln s+t\ln t+u\ln u$. When acting on the other polynomial terms in the momenta $\mathcal{D}$ is of order $1/\alpha'$. Therefore we can replace the $1/N$ by $\mathcal{D}$. Thus, $\mathcal{V}(X^{\mu N}_{\rm cl},P)=\mathcal{V}(X^{\mu 1}_{\rm cl}\mathcal{D},P)$, when acting on $A_{N-1}(P_i)$, to leading order $1/\alpha'$.

Using this trick we can write down linear relations between any two four-particle scattering amplitudes:

$$\textstyle\prod_i\mathcal{V}_{b_i}(X^\mu_{\rm cl}\mathcal{D},P_i)A_G^{a_i}(P_i)=\prod_i\mathcal{V}_{a_i}(X^\mu_{\rm cl}\mathcal{D},P_i)A_G^{b_i}(P_i)[1+O(\alpha')],\qquad(6)$$

which are independent of G! I can now argue that this relation should hold for the complete amplitude, in the limit $\alpha'\to\infty$, since it holds order by order in perturbation theory. This is a traditional method of proving the properties of quantum field theory (such as the operator-product expansion, symmetries, the renormalization group, etc.) by establishing their validity order by order in perturbation theory. It is not, however, without dangers, especially in the proof of asymptotic theorems. Nonetheless, we shall make this assumption. Then, since as $\alpha' \to \infty$ all the particles become massless, we can relate the scattering amplitudes for any set of particles. Indeed all the four-particle scattering amplitudes can be expressed in terms of say, the four-tachyon amplitude $A_{\text{tachyon}}(P_i)$:

$$A_{a_i}(P_i) = \prod_i \mathcal{V}_{a_i}(X^\mu_{\text{cl}}\mathcal{D}, P_i) A_{\text{tachyon}}(P_i)[1+O(\alpha')]. \tag{7}$$

This is a remarkable result, which hints at a very large and unusual symmetry of string theory that might be restored at high energies. Let us examine the simplest of these relations. Consider the scattering of the massless excitations of the closed, type-II, bosonic string. The physical states are given by the vertex operators $\mathcal{V}(X^\mu, P) = \epsilon^{\mu\nu}\partial X^\mu \bar{\partial} X^\nu$, where $P^\mu \epsilon^{\mu\nu} = \epsilon^{\mu\nu} P^\nu = 0$. The symmetric traceless part of $\epsilon^{\mu\nu}$ describes the graviton, the antisymmetric piece describes the antisymmetric tensor, and the trace describes the dilaton. In this nonsupersymmetric theory there is no *a priori* relation between the amplitudes of these different particles. However, the above relations will relate them all to each other. Since the vertices are all homogeneous of degree two in X^μ, we can dispense with the derivatives. We need only evaluate $\mathcal{V}$ for each particle by plugging (5) into $\mathcal{V}$. Upon doing this we note that the potential divergence when X^μ is taken at the puncture is removed by the physical-state conditions $P^\mu \epsilon^{\mu\nu}(P) = \epsilon^{\mu\nu}(P) P^\nu = 0$. Using these relations we find that

$$\partial X^\mu(a_1) = \bar{\partial} X^\mu(a_1) \propto Q^\mu = [P_2^\mu P_1 \cdot (P_3 - P_4) + P_3^\mu P_1 \cdot (P_4 - P_2) + P_4^\mu P_1 \cdot (P_2 - P_3)],$$

and thus that

$$\mathcal{V}_\epsilon(P_1) = Q^\mu \epsilon^{\mu\nu}(P_1) Q^\nu. \tag{8}$$

From this equation we can deduce the following. First, the amplitudes involving the antisymmetric tensor vanish relative to the others, as well as those involving polarizations of the graviton outside of the plane of scattering. Thus in the limit $\alpha' \to \infty$ the amplitudes of all the massless particles vanish, with the exception of the dilaton and the gravitons whose polarizations lie in plane of scattering. It is easy to calculate $\mathcal{V}$ for these, with the result that the nonvanishing amplitudes are all equal up to a momentum-independent constant. What symmetry could give rise to such relations?

Similar relations could surely be derived for the superstring and could also be extended to multiparticle amplitudes since most of the above story goes through for these. If this is the case then the full S matrix of the $\alpha' \to \infty$ limit of string theory (say for the heterotic string) could be expressed in terms of the dilaton S matrix. One could even contemplate constructing explicitly the $\alpha' \to \infty$ limit of the theory by plugging these relations into the unitarity equations (which according to the arguments of Ref. 1 might be valid for the limit $\alpha' \to \infty$, since the high-energy behavior is dominated by amplitudes in which all internal momentum transfers and energies are large) and using these to solve for the dilaton amplitudes and thereby the full $\alpha' = \infty$ theory. This approach is made quite complicated by the accumulation of an infinite number of particles at zero mass as $\alpha' \to \infty$.

The above relations connect amplitudes involving particles of different and arbitrarily high spin. If they are generated by a symmetry transformation of the $\alpha' \to \infty$ theory it must be one whose conserved charges have arbitrarily high spin. This would contradict the Coleman-Mandula theorem,[5] which limits the maximal spin of a conserved charge to be 1. Perhaps the Coleman-Mandula theorem is invalid for the $\alpha' \to \infty$ limit of string theory. One of the assumptions of the theorem is the *particle-finiteness* assumption,[5] which states that for any finite M there are only a finite number of particles with mass less than M. For $\alpha' = \infty$ there are an infinite number of massless particles, in which case the theorem need not apply.

It might also be the case that the Coleman-Mandula theorem is valid. The theorem states that if there exist higher-spin conserved charges then the S matrix equals the identity and the theory is trivial. This is because a higher-spin conserved charge, $Q_{\mu_1,\mu_2,\mu_3,\ldots}$, would take values equal to $\sum_i P^i_{\mu_1} P^i_{\mu_2} P^i_{\mu_3} \ldots$ on asymptotic states of spinless particles of momenta $P^1, P^2, P_3, \ldots$. The only way this can be conserved is if all the individual momenta are conserved, i.e., only forward and backward scattering is allowed. Then, if one accepts the usual analyticity of scattering amplitudes, no scattering at all is allowed (in more than two space-time dimensions). Perhaps this theorem is valid, the higher-spin symmetries do exist, and consequently the scattering amplitudes do all vanish as $\alpha' \to \infty$. This is certainly suggested by the exponential falloff of the scattering amplitudes, order by order in perturbation theory, as in (3), as $\alpha' \to \infty$. In this case the relations (6), although of the form 0/0, would still have much content.

It should be evident to the reader that the author has little idea as to the specific nature of these purported symmetries. If the above relations between the S-matrix elements are true, then they presumably do contain enough information for us to deduce the symmetry. One might, for example, attempt to construct an effective

Lagrangean, for some of the modes of the string, consistent with these relations and explore its symmetries. However, this might not be such an easy task given that the relations hint at a very unfamiliar and new kind of symmetry. Nonetheless I felt it useful to present these speculations, since any advance in the understanding of physics at the Planck length is of critical importance.

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