Generalization of the Fourier Transform: Implications for Inverse Scattering Theory

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An infinite number of ways are developed for representing a function in terms of the eigenfunctions of a three-dimensional scattering problem and simple known auxiliary functions. The utility of the new expansions, which generalize both the Fourier and Radon transforms, is shown by derivation of a new representation of the scatterer for the near- (far-) field inverse problem. Further, the scattering amplitude and potential are shown to be a generalized Fourier-transform pair.

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In many problems of theoretical physics, one is interested in the eigenfunctions of a differential operator. It is often useful to express an arbitrary function in terms of its projections onto these eigenfunctions. Probably the best-known example of this is the Fourier transform and its inverse. In this case, the differential operator is the Laplacian and the eigenfunctions are plane waves. Another example occurs in quantum scattering theory, which relies heavily on expansions in terms of the eigenfunctions of Schrödinger's equation.[1]

Eigenfunction expansions are also useful in work on the inverse scattering problem, which is the problem of recovering properties of a scatterer from scattering data. In the case of quantum inverse scattering, many investigators have used expansions in eigenfunctions of the Schrödinger operator.[1,2] In contrast, for classical wave theory (acoustics, electromagnetics, and elastodynamics) in three dimensions, eigenfunction expansions for the corresponding scattering operators have not been used. Researchers in inverse scattering have instead used the Fourier transform, which has led to their obtaining results that are only approximate. For example, it has been shown that in the case of weak scattering, the approximate (Born) scattering amplitude is a Fourier transform of the potential.[3]

In this paper, we consider a general wave equation that, in various special cases, reduces to the variable-velocity wave equation, the acoustic equation, and the Schrödinger equation. For this general wave equation, we exhibit an eigenfunction expansion. This eigenfunction expansion contains an auxiliary function, which, within certain constraints, can be chosen arbitrarily. Thus, we actually obtain an infinite number of related eigenfunction expansions for each wave equation. We show how to recover expansions for the Schrödinger, wave, and acoustic equations as special cases. These expansions lead to new generalizations of the Fourier and Radon transforms.[4] Finally, we show how to use the eigenfunction expansions in inverse scattering. In particular, we (1) derive new representations of the potential in terms of scattering data, and (2) show that the exact scattering amplitude is a generalized Fourier transform of the potential.

We consider first the acoustic equation[5]

$$[\nabla^2-(\nabla\rho/\rho)\cdot\nabla+\omega^2c^{-2}(\mathbf{x})]p(\omega,\mathbf{x})=0. \tag{1}$$

This equation governs wave propagation in an inhomogeneous fluid; here $\mathbf{x}$ is a coordinate in $\mathbb{R}^3$; ω, which denotes an angular frequency, is a real nonnegative scalar; p is the excess pressure; ρ, the density, and c, the local sound speed, are bounded positive functions that are bounded away from zero. We transform (1) into a more tractable equation by means of the transformation $\psi=\rho^{-1/2}p$. We thus obtain an equation of the form

$$[\nabla^2+\omega^2-V(\mathbf{x})\omega^2-q(\mathbf{x})]\psi(\omega,\mathbf{x})=0. \tag{2}$$

The solution $\psi(\omega,\mathbf{x})$ we call the wave field. We assume that (1) V and q are bounded and have compact support; (2) V and q have two continuous derivatives; (3) Eq. (2) has no bound states (i.e., solutions with rapid spatial falloff); and (4) q is nonnegative. These conditions are sufficient for our results but can surely be relaxed. We note that V is related to the local sound speed c by $V=1-c^{-2}$. Equation (2) is not only a useful form of (1), but it also reduces to the Schrödinger equation in the special case $V=0$.

We will be interested in scattering solutions that correspond to an incident plane wave, $\exp(i\omega\hat{\mathbf{e}}\cdot\mathbf{x})$, where $\hat{\mathbf{e}}$ is a unit vector denoting the direction of incidence. Two useful solutions are defined by the Lippmann-Schwinger equation[1,5]:

$$\psi^{\pm}(\omega,\hat{\mathbf{e}},\mathbf{x})=\exp(i\omega\hat{\mathbf{e}}\cdot\mathbf{x})+\int d^3\mathbf{y}\,G_0^{\pm}(\omega,\mathbf{x},\mathbf{y})[\omega^2V(\mathbf{y})+q(\mathbf{y})]\psi^{\pm}(\omega,\hat{\mathbf{e}},\mathbf{y}), \tag{3}$$

where

$$G_0^{\pm}(\omega,\mathbf{x},\mathbf{y}) = -(4\pi|\mathbf{x}-\mathbf{y}|)^{-1}\exp(\pm i\omega|\mathbf{x}-\mathbf{y}|).$$

We will also need the Green's functions, which are defined by

$$[\nabla^2+\omega^2-\omega^2V(\mathbf{x})-q(\mathbf{x})]G^{\pm}(\omega,\mathbf{x},\mathbf{y}) = \delta^3(\mathbf{x}-\mathbf{y}), \quad (4)$$

together with outgoing boundary conditions (plus) or incoming boundary conditions (minus). The quantity $G^+(\omega,\mathbf{x},\mathbf{y})$ can be thought of as the response at $\mathbf{x}$ to a point source at $\mathbf{y}$.

The causal properties[6] of the Green's functions are essential to our derivation. These are most easily described in terms of the "time-domain" Fourier transform

$$\hat{G}^{\pm}(t,\mathbf{x},\mathbf{y}) = (2\pi)^{-1}\int_{-\infty}^{\infty}d\omega\, e^{i\omega t}G^{\pm}(\omega,\mathbf{x},\mathbf{y}). \quad (5)$$

With our assumptions on $V(x)$ and $q(\mathbf{x})$, it can be shown[7] that $\hat{G}^+(t,\mathbf{x},\mathbf{y})=0$ for $t<|\mathbf{x}-\mathbf{y}|/c_m$ and $\hat{G}^-(t,\mathbf{x},\mathbf{y})=0$, $t>-|\mathbf{x}-\mathbf{y}|/c_m$. Here $c_m=\sup[c(\mathbf{x})]$ is an upper bound for the velocity of wave propagation.

Finally, we will need the scattering amplitude. It can be measured from the far field of ψ^+:

$$\psi^+(\omega,\hat{\mathbf{e}},\mathbf{x}) = \exp(i\omega\hat{\mathbf{e}}\cdot\mathbf{x})+A(\omega,\hat{\mathbf{e}}',\hat{\mathbf{e}})\exp(i\omega x)x^{-1}+o(x^{-1}). \quad (6)$$

Here $x=|\mathbf{x}|$ and $\hat{\mathbf{e}}=\mathbf{x}x^{-1}$. An explicit form for the scattering amplitude is obtained by expansion of the Lippmann-Schwinger equation for large x:

$$A(\omega,\hat{\mathbf{e}}',\hat{\mathbf{e}}) = -(4\pi)^{-1}\int d^3y \exp(-i\omega\hat{\mathbf{e}}'\cdot\mathbf{y})[\omega^2V(\mathbf{y})+q(\mathbf{y})]\psi^+(\omega,\hat{\mathbf{e}},\mathbf{y}). \quad (7)$$

We note that A satisfies the reciprocity relation[1] $A(\omega,\hat{\mathbf{e}}',\hat{\mathbf{e}})=A(\omega,-\hat{\mathbf{e}},-\hat{\mathbf{e}}')$. Incidentally, the scattering amplitude can also be obtained from "near-field" measurements.[8]

This completes our review of scattering theory; next we sketch a derivation of our basic result. The derivation starts with a formula obtained by Cheney *et al.*[9] by a Green's-theorem argument[10]:

$$G^+(\omega,\mathbf{x},\mathbf{y})-G^-(\omega,\mathbf{x},\mathbf{y}) = \int_{\partial\Omega}\left[G^+(\omega,\mathbf{z},\mathbf{y})\frac{\partial}{\partial n}G^-(\omega,\mathbf{z},\mathbf{x})-G^-(\omega,\mathbf{z},\mathbf{x})\frac{\partial}{\partial n}G^+(\omega,\mathbf{z},\mathbf{y})\right]dS_z, \quad (8)$$

where $\partial\Omega$ is a smooth surface enclosing the support of V and q, and n is the outer unit normal. Here $\mathbf{x}$ and $\mathbf{y}$ are either both inside Ω or both outside. Henceforth we will denote the right-hand side of (8) by $J(\omega,\mathbf{x},\mathbf{y})$. Next we take the Fourier transform of (8) to the time domain using (5). In doing this, we obtain $\hat{G}^+-\hat{G}^-$ on the left-hand side. But $\hat{G}^+$ and $\hat{G}^-$ have disjoint supports: If τ is any number whose magnitude is less than $|\mathbf{x}-\mathbf{y}|/c_m$, $\hat{G}^+(t,\mathbf{x},\mathbf{y})=0$ for $t<\tau$ and $\hat{G}(t,\mathbf{x},\mathbf{y})=0$ for $t>\tau$. Thus, if t is greater than τ, $\hat{G}^+(t,x,y)$ is equal to the Fourier transform of the right-hand side of (8) multiplied by the Heaviside function $H(t-\tau)$. We then take the Fourier transform back to the frequency domain, calculating the result at $\omega=0$. We thus have

$$G^+(0,\mathbf{x},\mathbf{y}) = (2\pi)^{-1}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}d\omega\exp(-i\omega t)H(t-\tau)J(\omega,\mathbf{x},\mathbf{y}). \quad (9)$$

We make the change of variables $s=t-\tau$; this brings out a factor of $\exp(-i\omega\tau)$. The s integral is then merely the Fourier transform of the Heaviside function, namely $\mathrm{P}(-i/\omega)+\pi\delta(\omega)$, P meaning principal value. The term containing the δ function vanishes because $J(0,\mathbf{x},\mathbf{y})$ vanishes, since $\mathbf{x}$ and $\mathbf{y}$ are either both inside Ω or both outside. Finally, we let ∇^2-q operate on the resulting equation. This gives us our main result:

$$\delta^3(\mathbf{x}-\mathbf{y}) = (2\pi)^{-1}[-\nabla^2+q(\mathbf{x})]\int_{-\infty}^{\infty}d\omega\,(i\omega)^{-1}e^{i\omega\tau(\mathbf{x},\mathbf{y})}J(\omega,\mathbf{x},\mathbf{y}), \quad (10)$$

where $-|\mathbf{x}-\mathbf{y}|/c_m\le\tau\le|\mathbf{x}-\mathbf{y}|/c_m$. Equation (10) and formulas based on it should be interpreted in distribution sense; in (10) the principal value is not needed because J vanishes at $\omega=0$. These matters are given a careful detailed treatment elsewhere.[11]

Equation (10) has a number of consequences, which correspond to different choices of τ and different configurations of $\mathbf{x}$, $\mathbf{y}$, and $\partial\Omega$. For example, if $\partial\Omega$ is taken to infinity, one can use the fact[10] that for large $|\mathbf{z}|$, $G^{\pm}(\omega,\mathbf{z},\mathbf{x})$ is asymptotic to

$$-(4\pi|z|)^{-1}\exp(\pm i\omega|\mathbf{z}|)\psi^{\pm}(\omega,\mp\hat{\mathbf{z}},\mathbf{x}),$$

where $\hat{\mathbf{z}}=\mathbf{z}/|z|$. Equation (10) becomes

$$\delta(\mathbf{x}-\mathbf{y}) = (16\pi^3)^{-1}[-\nabla^2+q(\mathbf{x})]\int_{-\infty}^{\infty}d\omega\, e^{i\omega\tau(\mathbf{x},\mathbf{y})}\int_{S^2}d^2\hat{z}\,\psi^{\pm}(\omega,\hat{\mathbf{z}},\mathbf{x})\psi^{\pm *}(\omega,\hat{\mathbf{z}},\mathbf{y}), \quad (11)$$

where S^2 is the unit sphere and the asterisk denotes complex conjugate. In obtaining (11) we have used the relations $\psi^+(-\omega,-\hat{\mathbf{z}},\mathbf{x})=\psi^-(\omega,\hat{\mathbf{z}},\mathbf{x})$ and $\psi^+(-\omega,\hat{\mathbf{z}},\mathbf{x})=\psi^{+*}(\omega,\hat{\mathbf{z}},\mathbf{x})$.

Equations (10) and (11) substantially generalize the eigenfunction expansions which are currently used for Schrödinger's equation, for the wave equation, and for the acoustic equation. In particular, Eq. (11) is a generalization of the inversion formula for the Fourier transform. This can be seen by an our taking $V=q=\tau=0$. Similarly, Eq. (11), when written in the time domain, is a generalization of the inversion formula for the Radon transform.

We believe that these expansions will be useful in both direct and inverse scattering theory as well as in more general problems of mathematical physics. As an example of the new expansions' usefulness, we derive below new representations of the potential in terms of scattering data and the wave field. Similar representations have been suggested as essential parts of self-consistent methods for solving the inverse problem. See Newton[12] for Schrödinger's equation and Rose and Cheney[13] for the wave equation.

First we note that Eq. (11) is a generalized eigenfunction expansion. It can be reduced to the usual eigenfunction expansions for the Schrödinger,[1] wave,[14] and acoustic equations[15] by making the choice $\tau=0$. Setting $\tau=\hat{\mathbf{e}}\cdot(\mathbf{x}-\mathbf{y})$ with $\hat{\mathbf{e}}\in S^2$ results in a different expansion, one that is useful in inverse scattering. To be specific, we will consider the variable-velocity wave equation ($q=0$). We make the further restriction that $c_m=1$. We then multiply both sides of (11) by $V(\mathbf{y})=1-c^{-2}(\mathbf{y})$ and integrate with respect to y. This yields

$$V(\mathbf{x})=(4\pi^2)^{-1}\nabla^2\int_{-\infty}^{\infty}d\omega\,\omega^{-2}\exp(-i\omega\hat{\mathbf{e}}\cdot\mathbf{x})\int d^2\hat{z}\,\psi^-(\omega,\hat{\mathbf{z}},\mathbf{x})A(\omega,\hat{\mathbf{z}},\hat{\mathbf{e}}), \qquad (12)$$

where A is the scattering amplitude [Eq. (7)]. Equation (12) shows that the sound speed and the scattering amplitude are transform pairs in the sense of (11).

Equation (12) was obtained by different methods in Ref. 13. A similar representation for the potential in the Schrödinger equation was obtained by Newton in Ref. 12. A representation for the speed in the wave equation in terms of near-field quantities was given in Ref. 9. All these representations are special cases of (10) and can be obtained simply by our making particular choices of the various parameters.

Equation (10) can also be used to obtain new results. We now derive a representation of V when either the transmitter or receiver is close to the scatterer; i.e., one or the other is in the near field but not both. Again we consider the wave equation with $c_m=1$. One sets $\tau(\mathbf{x},\mathbf{y})=|\mathbf{x}-\mathbf{p}|-|\mathbf{y}-\mathbf{p}|$ in (10); both sides of the resulting equation are multiplied by $V(\mathbf{y})\,(-4\pi|\mathbf{y}-\mathbf{p}|)$, and then integrated d^3y over $\mathbb{R}^3$. Next we use the equations $G^+=G_0^++\omega^2\int G_0^+VG^+$ and $G^{+\mathrm{sc}}=G^+-G_0^+$. Two representations for V are obtained; they correspond to different possible experiments. In the first $\mathbf{p}$ is chosen to be in the near field, while the boundary surface $\partial\Omega$ is taken to be the surface of an arbitrarily large ball centered about the scatterer. After using the far-field forms of $G^{\pm}$ one obtains

$$V(\mathbf{x})=-(4\pi^2)^{-1}|\mathbf{x}-\mathbf{p}|\nabla^2\int_{-\infty}^{\infty}d\omega\,\omega^{-2}e^{-i\omega|\mathbf{x}-\mathbf{p}|}\int_{S^2}d^2\hat{z}\,\psi^{+\mathrm{sc}}(\omega,\hat{\mathbf{z}},\mathbf{p})\psi^{+*}(\omega,\hat{\mathbf{z}},\mathbf{x}). \qquad (13)$$

For this representation of V the data are $\{\psi^{+\mathrm{sc}}(\omega,\hat{\mathbf{z}},\mathbf{p})$: all ω, all $\hat{\mathbf{z}}$ in S^2, $\mathbf{p}$ fixed$\}$. A second representation of V can be obtained by a slight modification of the derivation sketched. In this case one assumes that $\partial\Omega$ remains in the near field, while $|\mathbf{p}|$ is taken to become arbitrarily large (i.e., $\mathbf{p}$ is in the far field). The result is

$$V(\mathbf{x})=(8\pi^2)^{-1}\nabla^2\int_{-\infty}^{\infty}d\omega\,(i\omega^3)^{-1}e^{-i\omega\hat{\mathbf{p}}\cdot\mathbf{x}}\int_{\partial\Omega}ds_z\left\{\psi^{+\mathrm{sc}}(\omega,\hat{\mathbf{p}},\mathbf{z})\frac{\partial}{\partial n}G^-(\omega,\mathbf{z},\mathbf{x})-G^-(\omega,\mathbf{z},\mathbf{x})\frac{\partial\psi^+}{\partial n}(\omega,\hat{\mathbf{p}},\mathbf{z})\right\}. \qquad (14)$$

In this case the data consist of $\{\psi^{+\mathrm{sc}}(\omega,\hat{\mathbf{p}},\mathbf{z})$: all ω, all $\mathbf{z}$ on $\partial\Omega$, $\hat{\mathbf{p}}$ fixed$\}$.

Thus Eq. (10) can be used to obtain, in a simple and unified way, the representations of V given in Refs. 9, 12, and 13. Further, as we have just seen, (10) can easily be used to obtain representations of V for new experimental situations.

In summary, we have obtained an infinite set of generalized eigenfunction expansions. These expansions are appropriate for variable-velocity equations, such as the acoustic wave equation, as well as for Schrödinger's equation. They provide a natural generalization of the Fourier and Radon transform. Finally, they can easily be used to obtain new results for inverse scattering theory.

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