Numerical Studies of Enhanced Chiral Condensates

T. Appelquist, D. Carrier, L. C. R. Wijewardhana,[a] and W. Zheng
J. W. Gibbs Laboratory of Physics, Yale University, New Haven, Connecticut 06520
(Received 22 October 1987)

The results of a systematic numerical study of spontaneous chiral-symmetry breaking in gauge theories with slowly running couplings are summarized. For a range of such theories, the chiral condensate can be considerably enhanced with potentially important physical consequences in technicolor theories.

PACS numbers: 11.30.Qc, 11.30.Rd, 12.50.Lr

In a recent series of papers, two of us have proposed[1-3] that in asymptotically free theories with slowly running couplings, the chiral condensate $\langle\bar{\Psi}\Psi\rangle$ could be enhanced well above its naive value—roughly the cube of the confinement scale Λ of the theory. It was pointed out that this could be achieved without a substantial increase in the Goldstone-boson decay constant $F \simeq \Lambda$. In technicolor theories, this behavior could play a role in solving a long-standing problem.

The constraint of adequately suppressing the flavor-changing neutral currents in these theories has led to unacceptably low estimates for the ordinary-fermion masses and pseudo-Goldstone-boson masses. These masses, however, are directly proportional to the condensate $\langle\bar{\Psi}\Psi\rangle$. If the condensate can be enhanced, the above masses can be lifted into a more reasonable range without substantially affecting the masses of $W^{\pm}$ and Z^0.[4]

The purpose of the present Letter is to summarize the results of the first systematic numerical study of theories with slowly running couplings. The slow running is envisioned to arise from the only physical mechanism that we know of—the vacuum-polarization effects of the large number of fermions that typically inhabit these theories. Although sufficiently slow running is not generic to such theories (a modest fine tuning of the fermion number being required), it is not difficult to construct a long list[3] of potentially realistic, asymptotically free theories with the desired property.

The study being reported here has covered a range of the parameters describing a slowly running theory. An interesting limit that has been included in the study is that in which the coupling does not run at all above the chiral-symmetry-breaking scale. This limit effectively corresponds to the assumption of the existence of a nontrivial ultraviolet fixed point.[5-7] Since we know of no natural physical mechanism to produce such a behavior, we regard this limit as unphysical. It is nevertheless included since it provides an upper bound for the condensate-enhancement effect of asymptotically free theories.

The theories of interest will be governed by several parameters:

(1) A confinement scale Λ.

(2) A physical cutoff $M \gg \Lambda$. In technicolor theories, M is identified with the extended technicolor scale.

(3) The coefficient b of the lowest-order term in the β function. It comes purely from the gauge-field dynamics and is operative below the scale associated with fermion condensation. It is typically a number of order unity.

(4) The corresponding coefficient $b-\delta b$, operative above the fermion condensation scale. It contains the additional contribution $-\delta b$ due to fermion loops. In slowly running theories, $b-\delta b$ is typically less than $0.3b$.

Both b and δb can be regarded as effective one-loop parameters that include higher-order contributions. In many theories,[3] slow running is achieved only by inclusion of higher-order effects.

To produce order-of-magnitude estimates for the enhanced condensate, we will make use of the ladder approximation to the gap equation for the dynamical mass $\Sigma(p)$,

$$\Sigma(p)=\frac{3C_2}{2\pi}\left[\frac{\alpha(p)}{p^2}\int^p \frac{k^3dk\,\Sigma(k)}{k^2+\Sigma^2(k)}+\int_p^M \alpha(k)\frac{k\,dk\,\Sigma(k)}{k^2+\Sigma^2(k)}\right]. \quad (1)$$

Here, $\alpha(k)$ is the running coupling and C_2 is the Casimir operator of the fermion representation. Landau gauge is employed.[8] Above the scale $\mu \equiv 2\Sigma(0)$ of fermion condensation, the running coupling will be taken to be

$$\alpha(k)=\frac{\alpha(\mu)}{1+(b-\delta b)\alpha(\mu)\ln(k/\mu)}. \quad (2)$$

Since the gap equation is basically perturbative, it cannot be expected to govern chiral-symmetry breakdown completely. In particular, it is almost surely unreliable in detail for $k \le \mu$. We nevertheless parametrize this part of the equation by introducing a "confinement" scale Λ in terms of which $\mu \equiv 2\Sigma(0)$ is fixed. The running coupling $\alpha(k)$ is taken to reach some fixed value α_Λ at $k=\Lambda$ and to remain at this value for $k<\Lambda$. This value must be above the critical value $\alpha_c=\pi/3C_2$ required to trigger spontaneous symmetry breaking and give a nonzero $\Sigma(p)$.[9] For the numerical study being reported here, C_2 and therefore α_c are taken to be of order unity. If μ turns out to be above Λ, the coupling is taken to run according to $\alpha(k)=\alpha(\mu)/[1+b\alpha(\mu)\ln(k/\mu)]$ for $\Lambda \leq k \leq \mu$. All that is expected from this procedure is an order-of-magnitude estimate of μ/Λ. Since b is of order unity and since α_Λ/α_c is taken to be of order of (but somewhat larger than) unity, there are no small parameters in this momentum regime. The solution to the gap equation, therefore, produces what might be expected: $\mu/\Lambda \sim 1$. Since $F \simeq \mu$ is always a property of spontaneous chiral-symmetry breaking,[3] it then turns out that $F/\Lambda \sim 1$ for all the theories being considered here.[4] It is, of course, not this ratio but the relative size of the condensate and μ that is our main concern.

Another output of our treatment of the low-momentum part of the integral equation is the value of $\alpha(p)$ at $p \gtrsim \mu$. It is found, as expected, that $\alpha(p) \simeq \alpha_c$ for $p \gtrsim \mu$. Thus the combination $(b-\delta b)\alpha_c$ is a measure of the rate of running of the coupling for $p \gtrsim \mu$.

The gap equation is solved numerically with a self-consistent iterative procedure. The condensate can then be computed by our numerically evaluating the integral

$$\langle \bar{\Psi}\Psi \rangle_M = \frac{N}{2\pi^2} \int^M \frac{k^3 dk\, \Sigma(k)}{k^2+\Sigma^2(k)}, \tag{3}$$

where N is the dimensionality of the fermion representation.[10] An alternative expression can be obtained by our setting $p=M$ in the gap equation (1) and noting that the integral is the same as that appearing in Eq. (3). Thus

$$\langle \bar{\Psi}\Psi \rangle_M = \frac{NM^2\Sigma(M)}{3\pi C_2 \alpha(M)}. \tag{4}$$

The condensate is therefore determined by the values of $\Sigma(p)$ and $\alpha(p)$ at the cutoff. Another expression for $\langle \bar{\Psi}\Psi \rangle_M$ in terms of these quantities at the cutoff was derived by Bando *et al.*[11] by differentiation of Eq. (1). This yields

$$\langle \bar{\Psi}\Psi \rangle_M = \frac{N\Sigma'(M)}{3\pi C_2 [\alpha(M)/M^2]'}, \tag{5}$$

where the prime denotes differentiation with respect to M. We have used each of these expressions to evaluate $\langle \bar{\Psi}\Psi \rangle_M$.

The value of an ordinary-fermion mass in a technicolor theory is given by

$$m_f = (g_M^2/M^2)\langle \bar{\Psi}\Psi \rangle_M, \tag{6}$$

where g_M^2/M^2 is the strength of the relevant effective four-fermion interaction. The enhancement of $\langle \bar{\Psi}\Psi \rangle_M$ in a slowly running asymptotically free theory is due to the relatively slow initial fall of $\Sigma(k)$ before it takes on its final asymptotic form. To get a measure of the enhancement effect, we first note that without slow running the final asymptotic form sets in almost immediately beyond $p \simeq \mu$. It is $\Sigma(p) \sim (\Sigma_0^3/p^2)(\ln p)^{\gamma-1}$, where $\gamma = 3C_2/2\pi b$ and $\Sigma_0 \equiv \Sigma(0)$. Thus the value of $\langle \bar{\Psi}\Psi \rangle_M$ without enhancement can be estimated from Eq. (3) to be roughly of order $(N/2\pi^2)\Sigma_0^3$ times a numerical coefficient of order unity and a factor of $[\ln(M/\Sigma_0)]^\gamma$. If the theory is slowly running, producing enhancement, a good measure of the enhancement is the dimensionless integral

$$I_0 \equiv \frac{1}{\Sigma_0^3} \int \frac{k^3 dk\, \Sigma(k)}{k^2+\Sigma^2(k)}. \tag{7}$$

I_0 can equivalently be defined by use of the end-point expressions [Eqs. (4) and (5)].

Table I is a summary of our numerical results for I_0. It includes a range of slowly running asymptotically free theories as well as some normally running theories for comparison. The value of I_0 in the normal case lies in the range 5 to 10. This can be understood as arising partly from the $(\ln M)^\gamma$ factor mentioned above. For simplicity, a single value of $\alpha_c = \pi/3C_2$ and of b are used throughout the survey reported in the table. We have chosen $\alpha = 0.56$ and $b = 1.12$. The cutoff M is set equal

TABLE I. The condensate enhancement I_0 for a range of slowly running theories. I_0(int) is computed with Eq. (7) and I_0(end pt) is computed with Eq. (5) and divided by $(N/2\pi^2)\Sigma_0^3$. The low-momentum part of the gap equation [Eq. (1)] is approximated as described in the text leading as expected to $\Sigma_0/\Lambda \sim 1$, where Λ is an input confinement scale. Since the result $\alpha(\mu \equiv 2\Sigma_0)/\alpha_c \sim 1$ is also a feature of our treatment of the low-momentum part of Eq. (1), $(b-\delta b)\alpha_c$ is a measure of the rate of running of the coupling in the range $p > \mu$. We regard $(b-\delta b)\alpha_c \geq 0.5$ as normal running, and $(b-\delta b)\alpha_c \leq 0.3$ as slow running.

δb	$(b-\delta b)\alpha_c$	$\alpha(\mu \simeq 2\Sigma_0)$	I_0(int)	I_0(end pt.)
0.049	0.60	1.19	7.7	7.8
0.138	0.55	1.17	8.6	8.7
0.227	0.50	1.14	9.7	9.9
0.316	0.45	1.18	13.5	13.7
0.406	0.40	1.13	16.0	16.3
0.495	0.35	1.08	19.6	19.9
0.584	0.30	1.04	25	25
0.638	0.27	1.06	36	36
0.67	0.25	1.08	49	50
0.71	0.23	1.03	56	57
0.75	0.21	1.01	65	66
0.78	0.19	0.98	77	78
0.82	0.17	0.94	92	93
0.85	0.15	0.91	112	113
0.87	0.14	0.93	152	154
0.89	0.13	0.91	210	213
0.92	0.11	0.87	265	270
0.96	0.09	0.83	340	350
0.995	0.07	0.79	570	580
1.03	0.05	0.75	770	780
1.07	0.03	0.70	1350	1350
1.10	0.01	0.65	1800	1850
1.12	0.00	0.63	2750	2800

to $10^3\Lambda$.[12] We emphasize that these parameters, as well as those varying in Table I, are not intended to correspond to any specific theory. They are instead representative of a range of theories that run with varying degrees of slowness.

The qualitative reason for the large condensate enhancement in slowly running theories is that the dynamical mass $\Sigma(p)$ initially falls less rapidly than $(1/p^2)(\ln p)^{\gamma-1}$. When $\alpha(p) \simeq \alpha_c$, as is the case in the momentum range just above μ, $\Sigma(p)$ is expected to behave roughly like $1/p$.[2,3,7] In Refs. 2 and 3, a rough analytic estimate of I_0 was made by the assumption that this behavior persists all the way to $p \simeq \mu \exp\{1/[\alpha_\mu(b-\delta b)]\}$. It can be seen from Table I that this estimate is not bad for the slowly running range $0.3 \gtrsim (b-\delta b)\alpha_c \gtrsim 0.2$. For smaller values of $(b-\delta b)\alpha_c$, it begins to overestimate I_0. This is because the effect of the fixed cutoff M $(=10^3\Lambda)$ begins to become important. For very small values of $(b-\delta b)\alpha_c$, I_0 becomes of order 10^3, just what would be expected from Eq. (7), with the approximate $1/p$ behavior persisting nearly all the way to the cutoff M.

In Fig. 1, the actual power-law behavior of $\Sigma(p)$ is displayed for three examples from Table I. In a case of "normal" running, $(b-\delta b)\alpha_c = 0.6$ [Fig. 1(a)], the approximate $1/p^2$ behavior of $\Sigma(p)$ sets in very rapidly beyond $p=\mu$, and I_0 is correspondingly not so large. For the slowly running case $(b-\delta b)\alpha_c = 0.21$ [Fig. 1(b)], the power gradually increases from $\sim 1/p$ to $\sim 1/p^{1.75}$. The large value of I_0 is due to this effect. At the very end of the range, the rate of fall increases rather quickly to $1/p^2$. Finally, an example of very slow running is shown in Fig. 1(c). The initial fall of $\Sigma(p)$ is then even slower than p^{-1} leading to an even larger I_0. At the end of the range, the power fall again increases to approximately $1/p^2$.

The increase of the rate of fall of $\Sigma(p)$ to $1/p^2$ as p approaches the cutoff is a general feature of the gap equation (1). It is easy to see from this equation that in this limit, $d\ln\Sigma/d\ln p^2$ will approach $-1+(b-\delta b)\alpha(M)$. This is very close to -1 for each of the above three cases. Without the UV cutoff, the behavior of $\Sigma(p)$ would eventually approach $(1/p^2)(\ln p)^{\gamma-1}$. The cutoff, however, which should be regarded as a physical feature of technicolor theories, brings on the approximate $1/p^2$ behavior more abruptly. The fact that the initial fall of $\Sigma(p)$ in Fig. 1(c) is even slower than $1/p$ can be understood if it is remembered that when the running is neglected altogether, the solution of the linearized gap equation for $\alpha > \alpha_c$ is

$$\Sigma(p) = (C/p)\cos[\epsilon \ln(p/p_0)],$$

where $\epsilon = (\alpha/\alpha_c - 1)^{1/2}$. The instantaneous power behavior is then

$$-d\ln\Sigma/d\ln p = 1 + \epsilon \tan[\epsilon \ln(p/p_0)].$$

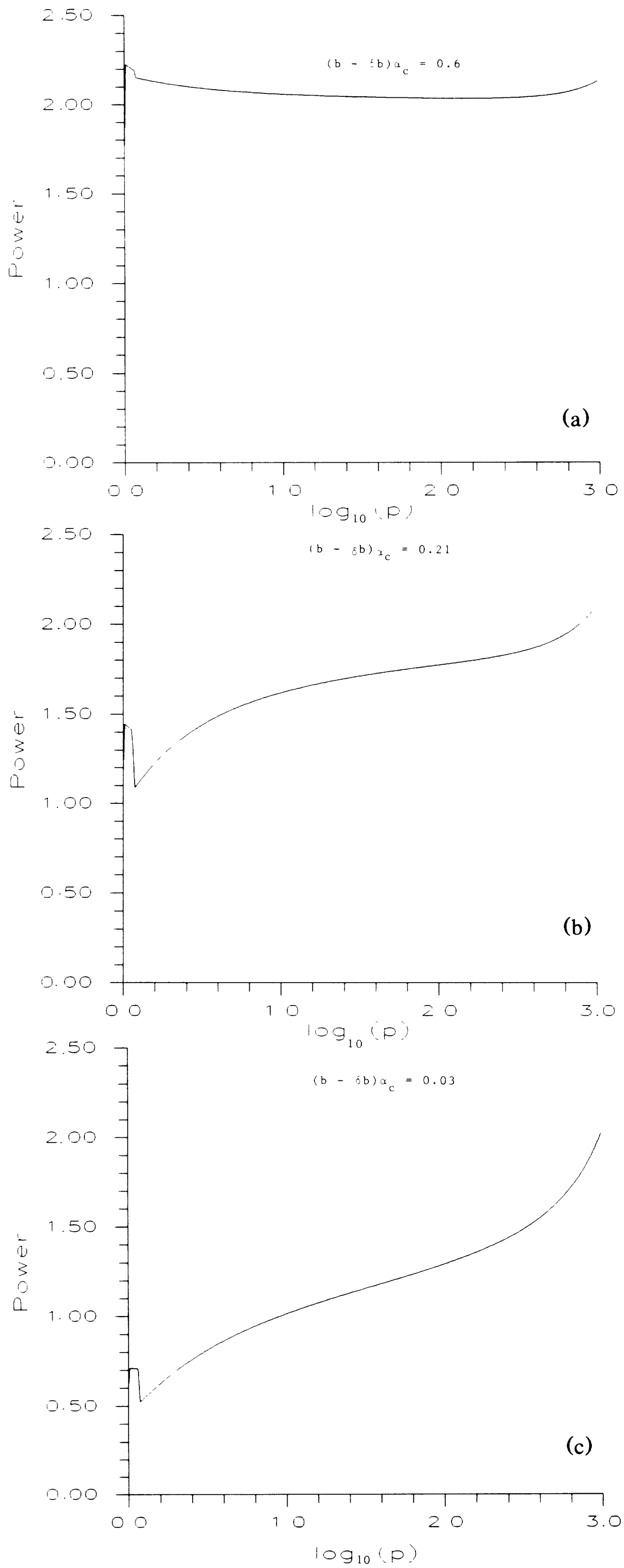

FIG. 1. The instantaneous power behavior $-d\ln\Sigma/d\ln p$ of $\Sigma(p)$ as a function of p. (a)-(c) are examples of normal, slow, and very slow running. p is in units of Λ.

Below p_0, the fall will be less rapid than $1/p$. It turns out numerically that $p_0 \approx 30\mu$ in the case of very slow running. For the slow-running case [Fig. 1(b)], $p_0 \approx \mu$ and the initial fall is already $1/p$ and faster.

It should be stressed again that the very smallest values of $(b-\delta b)\alpha_c$ are difficult to attain in realistic theories. There are, however, many theories[13] for which $0.25 \gtrsim (b-\delta b)\alpha_c \gtrsim 0.15$, leading to condensate-enhancement factors I_0 of up to 2 orders of magnitude and more. The numerical study being reported here makes it clear that this can and does happen, in agreement with the analytic estimates presented in Refs. 2 and 3. It also makes it clear that the arguments to the contrary presented by the authors of Ref. 11 are incorrect.

Finally, it is worth mentioning that we have made preliminary estimates of condensate enhancements for some theories that lose asymptotic freedom above the chiral condensation scale $\mu = \Sigma_0$. If the parameters are such that $\alpha(p)$ never gets far above α_c, μ can turn out to be on the order of the confinement scale, and therefore smaller than the cutoff M. Condensate enhancements can then be obtained that are much larger than those of Table I. A systematic study of this possibility will be reported elsewhere.

For any of the theories described here, it remains to be seen whether the large condensate-enhancement effect we have exhibited will play a role in a realistic theory of electroweak symmetry breaking.

This work was supported in part by the U.S. Department of Energy under Contracts No. DE-AC02-76ER03075 and No. FG-0284ER-40153.

[a]Present address: Department of Physics, University of Cincinnati, Cincinnati, OH 45221.

[1]T. Appelquist, D. Karabali, and L. C. R. Wijewardhana, Phys. Rev. Lett. **57**, 957 (1986).

[2]T. Appelquist and L. C. R. Wijewardhana, Phys. Rev. D **35**, 774 (1987).

[3]T. Appelquist and L. C. R. Wijewardhana, Phys. Rev. D **36**, 568 (1987).

[4]B. Holdom, Phys. Lett. **150B**, 301 (1985), examined slowly running theories to try to produce a hierarchy between the confinement and chiral breaking scales (between Λ and F in our notation). He also observed that these theories could lead to enhanced fermion masses while suppressing flavor-changing neutral currents.

[5]B. Holdom, Phys. Rev. D **24**, 1441 (1981); V. A. Miransky and P. I. Fomin, Fiz. Elem. Chastits At. Yadra **16**, 469 (1985) [Sov. J. Part. Nucl. **16**, 203 (1985)].

[6]W. A. Bardeen, C. N. Leung, and S. T. Love, Phys. Rev. Lett. **56**, 1230 (1986).

[7]K. Yamawaki, M. Bando, and K. Matumoto, Phys. Rev. Lett. **56**, 1335 (1986), and Phys. Lett. B **178**, 308 (1986).

[8]This form of the gap equation is valid only in gauges in which Z_2 is finite. Landau gauge in one loop and the generalized Landau gauge in higher loops satisfy this condition.

[9]R. Fukuda and T. Kugo, Nucl. Phys. **B117**, 250 (1976); M. Peskin, in *Recent Advances in Field Theory and Statistical Mechanics,* Proceedings of the Houches Summer School, Session XXXIX, edited by J. B. Zuber and R. Stora (North-Holland, Amsterdam, 1984).

[10]We have chosen to cut the integral off at M, assuming that beyond this scale the effective four-fermion vertex giving rise to the technifermion condensate in the computation of ordinary-fermion masses will open up leading to ultraviolet-convergent momentum integrals.

[11]M. Bando, T. Morazumi, H. So, and K. Yamawaki, Phys. Rev. Lett. **59**, 389 (1987).

[12]Without some special suppression mechanism, the cutoff must be at least this large to adequately suppress flavor-changing neutral currents.

[13]Some of these are tabulated in Ref. 3. The β-function computations there are taken to 3 orders in the loop expansion.