Exact Generating Function for Fully Directed Compact Lattice Animals

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The fully directed compact lattice-animal model on the square lattice is solved exactly. An explicit expression is obtained for the cluster-number generating function which has a complicated complex-plane singularity pattern including a natural boundary of essential singularities. However, the cluster numbers are controlled by a simple pole singularity and grow proportionally to λ^N, where N is the number of sites, and $\lambda \cong 2.66$.

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Models of lattice animals (connected clusters) have received much attention in recent years as simply systems with "geometrical" critical phenomena. Various aspects of the isotropic lattice animals have been reviewed, e.g., by Bovier, Froehlich, and Glaus,[1] Privman,[2] and Sykes.[3] Directed lattice animals have also been studied extensively, numerically and by analytic methods.[4-6] Several conjectured exact critical-exponent values and relations have been found for both types of animals.[4-12] When an additional *compactness* constraint is imposed with directedness, a new universality class is obtained.[13-15] Partially directed compact lattice-animal models on the square and triangular lattices can be solved exactly.[14-16] The generating functions for the numbers c_N of distinct N-site clusters,

$$G(z)=\sum_{N=1}^{\infty} c_N z^N, \tag{1}$$

turn out to be rational functions of z. The singularity nearest to the origin is a simple pole at $z_c=\lambda^{-1}<1$, where λ is model dependent. Thus, the generic lattice-animal cluster-number asymptotic form for large N,

$$c_N \approx AN^{-\theta}\lambda^N, \tag{2}$$

applies with the critical exponent $\theta=0$. For the fully directed square-lattice model, numerical studies[13,14] suggest that the *leading* singularity is similar ($\theta=0$). However, a recent exact calculation[17] for a related circle-stacking model[18] indicates that the generating function for the fully directed compact animals may have a far richer complex-plane structure.

In this work, we report exact results for the fully directed compact square-lattice animals. First, we define the model and present an exact expression for $G(z)$. We then discuss the complex-plane structure of this generating function, focusing on the new features not present in the previously studied partially directed models. Lastly, we outline our method of solution.

A fully directed lattice animal on the square lattice of unit spacing is defined as follows. Let the xy coordinate axes coincide with the principal lattice directions, with the origin at a lattice site. This origin site is always part of a cluster. The different *directed* N-site clusters are defined by all the possible selections of the remaining $N-1$ sites in such a way that each can be reached from the origin, by a walk of $+\hat{x}$ and $+\hat{y}$ steps, through other occupied cluster sites. Obviously, the directed axis for this problem is defined by the unit vector $(\hat{x}+\hat{y})/\sqrt{2}$. The compactness condition[13,14] is then added by the requirement that all cluster sites at a given "time," i.e., for fixed values of $x+y$, form a continuous chain of diagonal neighbors (there is no restriction in the case of a single site at a given $x+y$). Note that the sites at each time level (fixed $x+y>0$) are connected to the origin site (at $x=y=0$) via the sites in the preceding time level. The compactness condition has therefore an effect of suppressing branchings in a cluster. However, the terminology is somewhat misleading: The clusters can be quite sparse.

Our results for the generating function are summarized below:

$$G(z)=z\frac{(1-z^2)T_{12}(z)-z^3T_{13}(z)}{(1+z)(1-3z+z^2)T_{12}(z)-z^3(1-2z)T_{13}(z)+z^5T_{23}(z)}, \tag{3}$$

where

$$T_{ij}(z)=A_i(z)B_j(z)-A_j(z)B_i(z), \tag{4}$$

$$A_k(z)=\sum_{n=0}^{\infty} q_n^{-2}(z)z^{n(3n+2k+3)/2}, \tag{5}$$

$$B_k(z)=k+\sum_{n=1}^{\infty} q_n^{-2}(z)z^{n(3n+2k+3)/2}\left[k+n+2\sum_{m=1}^{n}\frac{1}{1-z^m}\right], \tag{6}$$

$$q_n(z)\equiv\prod_{m=1}^{n}(1-z^m), \quad \text{for } n\geq 1, \tag{7}$$

and $q_0(z) \equiv 1$. Thus $G(z)$ is rational in $A_k(z)$ and $B_k(z)$ which are represented by the infinite series (5) and (6). These series converge extremely rapidly for all $|z| < 1$. In fact, A_k and B_k are analytic in the unit circle, with a natural boundary-type essential singularity at $|z| = 1$. However, the singularity of $G(z)$ nearest to the origin is a pole due to a zero of the denominator of (3). The function $z/G(z)$ is plotted in Fig. 1. The pole (in G) is located at $z_c = 0.375677\ldots$, corresponding to

$$\lambda = 2.6618579439\ldots. \tag{8}$$

In fact, $G(z)$ has an infinite sequence of poles and zeros in $(z_c, 1)$, outside the range of Fig. 1, accumulating at $z = 1^-$. For the *partial directed* square-lattice compact animals, the generating function is[14] a rational function of z; the "physical" singularity is a simple pole; however, there are no other interesting complex-plane features. Although the concept of *universality* refers to the leading singularity only, the global difference between the two models is surprising. A related phenomenon may be that in many lattice-spin models the behavior on certain loci in the parameter space is particularly simple to allow analytic treatment.[19] Finally, let us mention that certain cluster models have been considered in the literature[5,18] for which the essential singularity at $z = 1$ is the dominant one, and (2) no longer applies.

The derivation of the above results involves several steps. First, we consider animals rooted at k sites,[4,13,14] at

$$(x,y) = (0,k-1),(1,k-2),\ldots,(k-1,0), \tag{9}$$

i.e., all the remaining $N-k$ sites must be reachable by the $+\hat{x},+\hat{y}$ steps, from *at least one* root site. Thus, the originally defined problem corresponds to $k = 1$. Let $c_{N,k}$ denote the number of distinct k-root N-site animals, and define

$$F_k(z) = \sum_{N=k}^{\infty} c_{N,k} z^{N-k}, \tag{10}$$

where $c_{k,k} = 1$. These generating functions satisfy the recurrence relations[13,14]

$$F_k(z) = 1 + \sum_{m=1}^{k+1} (k-m+2) z^m F_m(z). \tag{11}$$

These can be reduced to

$$F_{k+2}(z) - 2F_{k+1}(z) + F_k(z) = z^{k+3} F_{k+3}(z), \quad k \geq 1, \tag{12}$$

with the boundary conditions

$$F_1 = 1 + 2zF_1 + z^2 F_2, \qquad F_2 = 1 + 3zF_1 + 2z^2 F_2 + z^3 F_3. \tag{13}$$

The functions $A_k(z)$ and $B_k(z)$ in (5) and (6) are, in fact, the two physically acceptable, i.e., regular at $z = 0$, solutions of the third-order difference equation (12). (The third solution is singular at $z = 0$.) Thus, $F_k(z) = \alpha(z) A_k(z) + \beta(z) B_k(z)$, where α, β must be determined by (13). The result for the generating function of the original problem,

$$G(z) \equiv z F_1(z), \tag{14}$$

is given by (3), with (4)-(7).

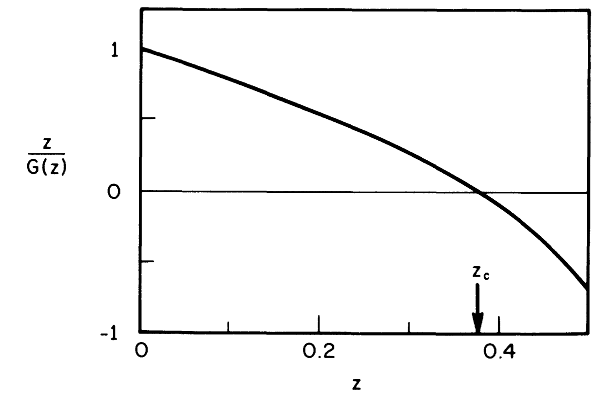

FIG. 1. Plot of $z/G(z)$, covering the physically relevant region $0 \leq z < z_c \cong 0.375677$.

One solution of (12) can be determined by consideration of a more general problem,

$$F_{k+2} - 2F_{k+1} + F_k = z^{k+m} F_{k+l}, \quad l \geq 0. \tag{15}$$

For $l = 0,1,2$ this is a second-order equation which can be analyzed by the continued-fraction method.[20] The $l = 0,1,2$ results can then be represented as infinite sums, based on the classical identities due to Ramanujan; see Bhargava and co-workers.[21,22] This leads to an *Ansatz* for general l,

$$F_k(z) = \sum_{n=0}^{\infty} q_n^{-2} z^{kn} f_n, \tag{16}$$

which, upon substitution in (15), yields after some algebra

$$f_n = z^{nm + (n-1)nl/2}. \tag{17}$$

For $l,m = 3$ we thus obtain the solution A_k for (15). This solution was *guessed* already in Ref. 17. However, the present systematic approach via the *Ansatz* (16) allows a full solution of the problem since it also applies to certain *inhomogeneous* equations of the type (15). Indeed, for large k and *fixed* $0 \leq z < 1$, the two "physical" solutions of (15) behave according to $A_k \to 1$, while $B_k \propto k$. This suggests the form

$$B_k(z) = k A_k(z) + C_k(z). \tag{18}$$

After some algebra, we obtain the following inhomogeneous equation for C_k:

$$C_{k+2}-2C_{k+1}+C_k=z^{k+m}C_{k+l}+(l-2)A_{k+2}-2(l-1)A_{k+1}+lA_k. \tag{19}$$

A long calculation utilizing an *Ansatz* similar to (16) for $C_k(z)$ in (19), finally yields

$$B_k(z)=k+\sum_{n=1}^{\infty} q_n^{-2} z^{n[l(n-1)+2(k+m)]/2}\left[k+n(l-2)+2\sum_{p=1}^{n}(1-z^p)^{-1}\right], \tag{20}$$

which reduces to (6) for $l,m=3$.

In summary, we presented a novel analytic solution for a lattice-animal model. The *fully directed* model solved has a new intricate pattern of the generating-function singularities in the complex plane. The leading simple-pole singularity is the same as for the *partially directed* models so that universality applies. Identification of universality classes for various connected cluster models is crucial in the scaling theory of branched polymer conformations as reviewed, e.g., by de Gennes.[23]

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