Macroscopic Approach to Universal Conductance Fluctuations in Disordered Metals

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Recently, a macroscopic theory of N-channel disordered conductors showed that the statistical distribution of the transfer matrix for a system of length L evolves with L according to a diffusion equation in N dimensions. It is proved here that the recently observed universal conductance fluctuations in normal metals at very low temperatures are a rigorous consequence of that diffusion equation, in the regime in which $L \gg$ (mean free path) and $N \gg 1$. The value found for the fluctuation coincides with the one obtained from elaborate microscopic calculations.

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In recent years, experiments[1] have shown fluctuations in the conductance of normal metals in the highly conducting diffusive regime at low enough temperatures that the inelastic mean free path is larger than the spatial dimensions of the system. One of the most intriguing results is that the variance of the conductance g (measured in units of e^2/h) is of order unity, i.e., $\mathrm{var}g \simeq 1$, and does not depend on the size of the sample or its average conductance: This phenomenon is called the *universal conductance fluctuation.*[2]

The above striking result has been studied theoretically by a number of authors[2,3] using a perturbative treatment, or numerical simulations, or, very recently, by proposing an interesting connection between the statistical properties of the transfer matrix and the familiar ensembles of random-matrix theory.[4,5]

A theory of multichannel disordered conductors was presented by Mello, Pereyra, and Kumar[6]; it was named "macroscopic" because, just like Refs. 4 and 5, it deals with the statistical distribution of the transfer matrix for the *full* conductor. The theory is based on the general properties of the scattering system: flux conservation, time-reversal invariance (in the absence of a magnetic field), and the appropriate combination law when two wires are put together. The distribution associated with systems of very small length is selected on the basis of a *maximum-entropy* criterion; the combination law allows then to find rigorously—and that was the central result of Ref. 6—the "evolution" of that distribution with the length L: It turns out to be governed by a Fokker-Planck or diffusion equation in N dimensions, where N is the number of channels. It is the purpose of the present Letter to prove that the universal conductance fluctuations are a *rigorous* consequence of such a diffusion equation, in the regime in which $L \gg l$ (l being the elastic mean free path) and $N \gg 1$.

I first sketch some of the relevant results of Ref. 6. The notation of Ref. 6 will be occasionally altered for convenience.

In the scattering approach, the disordered system (a piece of wire) is sandwiched between two perfect leads, where the scattering states, at the Fermi energy, define the N channels; each channel can carry two waves, propagating in opposite directions. The wave function outside the scattering system is thus specified by a $2N$-component vector, whose first N components are the amplitudes of the waves traveling to the right, and the remaining components are the N amplitudes traveling to the left. By definition, the $2N \times 2N$ transfer matrix relates the vector on the right with that on the left of the system. Flux conservation and time-reversal invariance require $R\Sigma_z R^\dagger = \Sigma_z$, and $R^* - \Sigma_x R \Sigma_x$, where

$$\Sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

have the structure of Pauli matrices, 1 indicating the $N \times N$ unit matrix. With these restrictions, an R matrix has $N(2N+1)$ independent parameters and can always be represented in the form

$$R = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \begin{pmatrix} (1+\lambda)^{1/2} & \sqrt{\lambda} \\ \sqrt{\lambda} & (1+\lambda)^{1/2} \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & v^* \end{pmatrix}, \qquad (1)$$

where u, v are arbitrary $N \times N$ unitary matrices and λ is a real, diagonal matrix with N arbitrary positive elements $\lambda_1, \ldots, \lambda_N$. As an application of the above parametrization, I mention that when the channels are fed with N incoherent unit fluxes, the total transmission coefficient $T = \sum_a T_a$ into all channels is given by $T = \sum_a (1+\lambda_a)^{-1}$. This is a very important quantity, since in the metallic regime, where the sample dimensions are much greater than l and each $T_a \ll 1$, the conductance g (including spin) is given by[7-9] $g = 2T$.

A collection or ensemble of random conductors of length L is described by an ensemble of R matrices, whose differential probability $dP_L(R) = p_L(R) d\mu(R)$ depends parametrically upon L. Here $d\mu(R)$, the *invariant measure* associated with the group of R's, is given by[5,6] $J(\lambda) \prod_a d\lambda_a \, d\mu(u) d\mu(v)$, where $J(\lambda) = \prod_{a<b} |\lambda_a - \lambda_b|$, and $d\mu(u)$ [and $d\mu(v)$] is the invariant measure of the unitary group $U(N)$.

The probability density $p_L(R)$ must satisfy an *important combination law.* If we put together two wires of lengths L and δL, with probability densities p_L and $p_{\delta L}$, the resulting probability density is given by the convolution $p^{(R)}_{L+\delta L} = p_L \circ p_{\delta L}$. The "building block" $p_{\delta L}$ is then constructed by our drawing an analogy with a well-known problem. The structure of the above combination requirement resembles that of the Smoluchowski equation, as is used, for instance, in connection with the theory of Brownian motion: The role of length is played by time in the Brownian problem, while our building block has its equivalent in the transition probability $p_{\delta t}(\delta \mathbf{u})$. The standard assumption for $p_{\delta t}(\delta \mathbf{u})$ (i.e., a Gaussian) is just the distribution of maximum entropy,[10] constrained by a fixed value of $\langle \delta \mathbf{u} \cdot \delta \mathbf{u} \rangle / 6\delta t = D$, the diffusion coefficient. Similarly, in the random conductor problem, Ref. 6 chooses for $p_{\delta L}$ the distribution of maximum entropy, constrained by a fixed value of the average reflection probability per unit length, which is the inverse *mean free path* l^{-1}. In the Brownian problem one obtains, from the Smoluchowski equation, a diffusion equation. Similarly, in the present problem, when the *Ansatz* for $p_{\delta L}$ is introduced in the combination law, one finds, for the joint probability density $w_L(\boldsymbol{\lambda}) \equiv p_L(\boldsymbol{\lambda}) J(\boldsymbol{\lambda})$ of $\lambda_1, \ldots, \lambda_N$, the Fokker-Planck or diffusion equation (with $s \equiv L/l$)

$$\frac{\partial w_s(\boldsymbol{\lambda})}{\partial s} = \frac{2}{N+1} \sum_{a=1}^{N} \frac{\partial}{\partial \lambda_a} \left[\lambda_a (1+\lambda_a) J(\boldsymbol{\lambda}) \frac{\partial}{\partial \lambda_a} \frac{w_s(\boldsymbol{\lambda})}{J(\boldsymbol{\lambda})} \right], \tag{2}$$

to be solved with the initial condition $w_0(\boldsymbol{\lambda}) = \delta(\boldsymbol{\lambda})$.

From Eq. (2) one can obtain, in principle, the evolution of the expectation value of any quantity of interest. For instance, multiplying both sides of (2) by T^p and integrating, one obtains the evolution equation for the pth moment of the total transmission factor as

$$(N+1)\partial_s \langle T^p \rangle_s = \langle -pT^{p+1} - pT^{p-1}T_2 + 2p(p-1)T^{p-2}(T_2 - T_3) \rangle_s. \tag{3}$$

Here, $\langle \ \rangle_s$ indicates an average performed with the probability density $w_s(\boldsymbol{\lambda})$ of Eq. (2). I have also used the notation $T_k \equiv \sum_a (1+\lambda_a)^{-k}$.

We notice that on the right-hand side of (3) there appear quantities other than $\langle T^q \rangle_s$, so that their evolution equations are needed as well. Since we are interested in the limit $N \gg 1$, we shall seek the solution to the coupled equations as a series in decreasing powers of N. Therefore, even though the exact evolution equations for the quantities appearing in (3) can be obtained exactly from (2), I quote here only those terms that are relevant to the analysis to be carried out below. One finds

$$(N+1)\partial_s \langle T^{p-1}T_2 \rangle_s = \langle 2T^{p+1} - (p+3)T^pT_2 + 2T^{p-1}T_2 - 4T^{p-1}T_3 - pT^{p-2}T_2^2 \rangle_s + O(N^{p-1}), \tag{4}$$

$$(N+1)\partial_s \langle T^{p-1}T_3 \rangle_s = \langle -(p+5)T^pT_3 + 6T^pT_2 - 3T^{p-1}T_2^2 \rangle_s + O(N^p), \tag{5}$$

$$(N+1)\partial_s \langle T^{p-2}T_2^2 \rangle_s = \langle -(p+7)T^pT_2^2 + 4T^{p+1}T_2 \rangle_s + O(N^{p+1}). \tag{6}$$

As I mentioned earlier, I propose the series expansion (with s a fixed number, independent of N)

$$\langle T^p \rangle_s = N^p f_{p0}(s) + N^{p-1} f_{p1}(s) + N^{p-2} f_{p2}(s) + \cdots, \tag{7}$$

and similarly for $\langle T^{p-1}T_2 \rangle_s$, $\langle T^{p-1}T_3 \rangle_s$, and $\langle T^{p-2}T_2^2 \rangle_s$, with $f_{pm}(s)$ replaced by $g_{pm}(s)$, $h_{pm}(s)$, and $l_{pm}(s)$, respectively. I introduce these series expansions in Eqs. (3)–(6) above and equate the coefficients of the various powers of N. In the first step we obtain

$$\partial_s f_{p0}(s) = -p f_{p+1,0}(s), \tag{8}$$

just as was found in Ref. 6. We see that the $f_{p0}(s)$ satisfy a closed, albeit infinite, set of coupled equations. The solution satisfying $f_{p0}(0) = 1$ is given by

$$f_{p0}(s) = (1+s)^{-p}. \tag{9}$$

For $p = 1$, (9) gives the leading term (in powers of N) of the average conductance as $\langle g \rangle_s = 2N(1+s)^{-1}$ and, for $s \gg 1$ (i.e., $L \gg l$),

$$\langle g \rangle_s \approx 2N/s = 2Nl/L. \tag{10}$$

Equation (10) is Ohm's law, when $N \sim (k_F W)^{d-1}$, W being the transverse dimension of the wire and d the dimensionality.

Equating the coefficients of the next power of N and using (9), one obtains a closed set of equations for $f_{p1}(s)$ and $g_{p0}(s)$ that can again be solved. Next, one obtains a closed set of equations for f_{p2}, g_{p1}, h_{p0}, and l_{p0}. Substituting $f_{p0}(s)$, $f_{p1}(s)$, and $f_{p2}(s)$ in Eq. (7), one obtains, for the pth moment of the transmission factor,

$$\langle T^p \rangle_s = \frac{N^p}{(1+s)^p} - \frac{ps^3}{3(1+s)^{p+2}} N^{p-1} + \frac{p}{90(1+s)^{p+6}} [(11p-9)s^8 + \cdots] N^{p-2} + \cdots. \tag{11}$$

In Eq. (11), the square bracket contains a polynomial of the eighth order in s, of which I only quote the highest power.

From (11) we now calculate var$T$: The first two terms in the expansion *cancel exactly,* giving, for $s \gg 1$, the leading term

$$\mathrm{var}T = \tfrac{2}{15} + \cdots . \tag{12}$$

I have thus proved *rigorously* that the leading term occurring in var$T$ is *independent* of the number N of channels (determined by the cross section of the wire), the length L of the conductor, and the mean free path l. The rms conductance $g = 2T$ is thus a *universal* number; i.e.,

$$\mathrm{rms}g = \sqrt{8/15} = 0.730\ldots . \tag{13}$$

This is precisely the value found in Ref. 2 in the quasi 1D case, with use of microscopic Green's function techniques. The statement made at the beginning of this Letter is thus proved.

I conclude that, as far as the above studied problem goes, the conceptually very simple *macroscopic* model used here appears to contain the same physical information as detailed, *microscopic* calculations.

I remark that the existence of *universal* conductance fluctuations can be viewed as a general consequence of the spectral rigidity of the transfer matrices [just as in standard random-matrix ensembles (see Brody *et al.*[11] for a review)], which was first suggested in Ref. 4 and explicitly shown in Ref. 5. It is clear, though, that more specific information on the ensemble is required in order to find a detailed quantitative result for rms$g$. In Ref. 6 an ensemble is proposed, based on a maximum-entropy hypothesis and characterized by one fundamental microscopic property (the mean free path l), which does permit such a detailed calculation, as shown in the present paper: The result so obtained was seen to be successful for quasi 1D systems. To understand further the range of validity of the model, it is thus in order to make a few additional comments.

It is significant that the diffusion equation (2), obtained through a maximum-entropy *Ansatz,* is indeed common to a whole class of microscopic models.[12] Suppose that we build up the conductor by adding n scattering units, each with a probability distribution $p_1(R)$, and ask for the resulting distribution for the full system. If we assume p_1 to be "isotropic," i.e., dependent only upon the λ's of Eq. (1), then, as $n \to \infty$ and in the weak-scattering regime, one finds, for the resulting distribution, precisely the diffusion equation (2); the only property of p_1 that occurs in the end is the parameter l, other characteristics of p_1 being washed out in the limiting process. I refer to this universal result as a generalized "central-limit theorem." This is a very satisfactory property of the model.

This isotropy assumption seems very reasonable and is mathematically very convenient, because it has the important property of being conserved under successive convolutions.[6] However, I want to make it clear that it cannot hold generally. Indeed, it is not surprising that isotropy breaks down for systems that are short compared with their width: For such a geometry, the assumption that the matrices u and v [Eq. (1)] couple one given channel equally likely to all other channels does not seem justified.[12,13] How to generalize the property of isotropy is still not clear to me and will certainly be an important task for the future.

Another limitation of the model, which is perhaps related with the discussion of the previous paragraph, is that there is no explicit reference to the dimensionality of the system, except through the number N of channels.[12]

Finally, I remark again that the whole analysis was made under the assumption of time-reversal invariance, thus implying the absence of a magnetic field. Relaxing this condition will also be an important generalization for the future.[14]

Before closing, I wish to remark that a maximum-entropy hypothesis has been used successfully in other areas of physics, like statistical mechanics[15] and molecular,[10] chemical,[10] and nuclear physics[16] (see also Levine and Tribus[17]). In Ref. 16, for instance, excellent quantitative agreement was found between macroscopic and very detailed microscopic calculations in the field of statistical nuclear reactions, showing once again that both contain the same physical information. It has been our experience that such a situation arises when there is a generalized central-limit theorem behind the scene that appears to govern the behavior of the system. There thus seems to be a close connection between maximum entropy and a generalized central-limit theorem whose nature is very important to explore further.

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