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COUPLED *S*- AND *P*-WAVE SOLUTIONS FOR PION-PION SCATTERING

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In a recent paper by Moffat,[1] a set of coupled equations was derived for the inverse partial-wave amplitudes on the basis of the partial-wave analysis of Chew and Mandelstam.[2] We present here the results of a numerical iteration of these equations when coupled S and P waves are treated, and all but two-pion exchange mechanisms are neglected. Up to and including P waves there occur two S-wave subtraction constants, $a_0^I = A_0^I(-\frac{2}{3})$, one P-wave subtraction constant, $a_1^1 = A_1^1(-\frac{2}{3})$, and a constant ξ_1, generated by the P-wave threshold behavior.

By using crossing symmetry, Chew and Mandelstam[2] have deduced that so long as D and higher waves are small the following five conditions hold at $\nu = -\frac{2}{3}$:

$$\tfrac{1}{5}a_0^0 = \tfrac{1}{2}a_0^2 = -\lambda, \qquad (1)$$

$$\tfrac{2}{9}a_0^{2\prime} = -\tfrac{1}{9}a_0^{0\prime} = a_1^{1\prime}, \qquad (2)$$

$$a_0^{0\prime\prime} - \tfrac{5}{2}a_0^{2\prime\prime} = 27a_1^1 + 18a_1^{1\prime}. \qquad (3)$$

All other derivative conditions which may be deduced at the symmetry point depend mainly on the omitted higher partial waves.

The system of equations for the inverse amplitude was programmed for an iterative solution on the CERN mercury computer, supplying λ, the effective pion-pion coupling constant, as a parameter. The constants a_1^1 and ξ_1 are determined by imposing the two crossing-symmetry conditions (2). The solutions, as functions of the single parameter λ, are used to determine the quantities entering (3). The extent to which the condition (3) is satisfied is a test of the consistency of the theory, and, as we shall see, this does not impose any limitations on the values of λ employed. If the solutions are to be consistent with the Mandelstam representation, no complex poles may occur in $A_l(\nu)$. It has been verified that no such poles occur for the solutions presented in this Letter.

Seven or eight interation cycles were found to be sufficient to determine the inverse amplitudes to better than 1%, when starting from given values of ξ_1 and a_1^1. By varying ξ_1 and a_1^1 in a systematic manner until Eqs. (2) were satisfied to 1-2%, these constants and the calculated phase shifts for a given λ were found to be of the same order of accuracy. In a preliminary calculation[3] the conditions (2) were only satisfied to 15%, but this accuracy has been found to be insufficient to obtain a quantitatively correct set of phase shifts.

The numerical calculations have shown clearly that the range of λ is limited. For each negative λ (corresponding to attractive S-wave forces) with $|\lambda| \leq 0.5$, a single positive value of ξ_1 can be found so that conditions (2) are satisfied. These solutions are characterized by a very small P-wave phase shift and correspond to those computed by Chew, Mandelstam, and Noyes.[4] However, these solutions with positive ξ_1 completely fail to satisfy condition (3) and do not, therefore, represent cases of physical interest. For negative values of λ, other solutions are found having a negative value of ξ_1. For values of $|\lambda| \geq 0.45$ a bound state in the $I=0$ state occurs ($\cot\delta_0^0$ becomes negative near zero energy), but

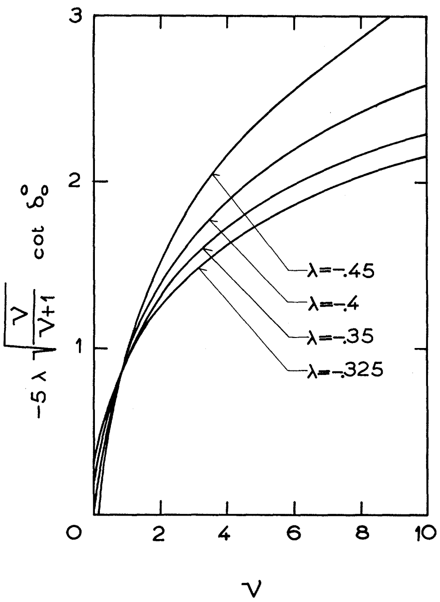

FIG. 1. $(-5\lambda)[\nu/(\nu+1)]^{1/2}\cot\delta_0{}^0$ as a function of ν, for different values of λ. Note that $\cot\delta_0{}^0$ for $\lambda=-0.45$ becomes negative near $\nu=0$.

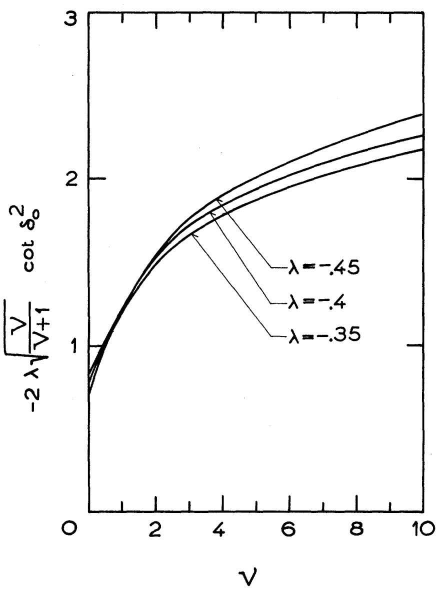

FIG. 2. $(-2\lambda)[\nu/(\nu+1)]^{1/2}\cot\delta_0{}^2$ as a function of ν, for different values of λ. The curve for $\lambda=-0.325$ is close to that for $\lambda=-0.35$ and is not shown in the interests of clarity.

for smaller values of $|\lambda|$ this disappears and the solutions are characterized by large P-wave phase shifts. The solutions with negative ξ_1 are all found to satisfy the second derivative condition (3), and therefore represent acceptable physical solutions.

Solutions of the coupled equations for positive λ (repulsive S-wave forces) with $\lambda > 0.25$ fail to satisfy conditions (2). For $\lambda < 0.25$ the iterative procedure adopted becomes unstable and we have been as yet unable to determine whether solutions exist. The calculated values of $-5\lambda[\nu/(\nu+1)]^{1/2}\cot\delta_0{}^0$, $-2\lambda[\nu/(\nu+1)]^{1/2}\cot\delta_0{}^2$, and $[\nu^3/(\nu+1)]^{1/2}\cot\delta_1{}^1$ are displayed in Figs. 1, 2, and 3 for $-0.45 \leq \lambda \leq -0.325$, and the corresponding values of $a_1{}^1$ and ξ_1 are shown in Table I. Results for smaller values of $|\lambda|$ and full details of the calculations will be published elsewhere. It should be noted that the $I=0$ phase shifts are much larger than $I=2$ phase shifts at low energies, and that the S-wave scattering lengths become smaller as the resonance position moves towards smaller energies. (This can be observed for the S-wave scattering lengths α_0, α_2, which have been included in Table I.) For $|\lambda| < 0.45$, $\cot\delta_1{}^1$ passes through zero from positive to negative values, this giving rise to a P-wave resonant cross section. The resonance positions ν_R and the

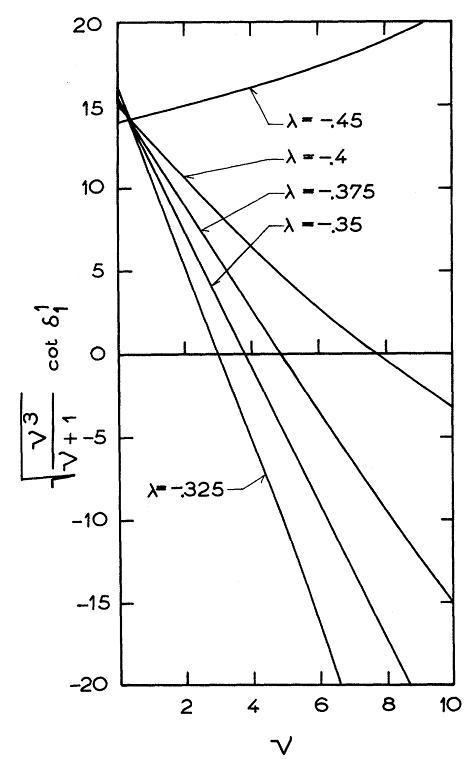

FIG. 3. $[\nu^3/(\nu+1)]^{1/2}\cot\delta_1{}^1$ as a function of ν, for different values of λ.

Table I. Calculated values of a_1^1, ξ_1, resonance parameters ν_R, Γ, and S-wave scattering lengths α_0, α_2.

λ	a_1^1	ξ_1	ν_R	Γ	α_0	α_2
-0.45	-0.0529	-21.2	...	...	-68.0	+1.24
-0.40	-0.0442	-22.5	7.7	6.0	+33.7	1.05
-0.375	-0.0413	-22.9	4.8(5)	2.5	16.4	0.96
-0.35	-0.0397	-23.4	3.8	1.7(5)	10.8	0.87
-0.325	-0.0373	-24.1	3.0	1.0	7.5	0.79

corresponding total widths Γ are also included in Table I [the total width Γ is related to the reduced width γ by $\Gamma = (\nu_R^3/(\nu_R+1))^{1/2}\gamma$].

The P-wave resonance suggested by Frazer and Fulco[5] was at $\nu_R = 1.5$ with a reduced width $\gamma = 0.4$, but more recently Bowcock et al.[6] and Frautschi[7] have found that on the basis of π-N scattering the resonance position ν_R should be further out at $\nu_R = 4.6$ with a reduced width $\gamma = 0.2$ and $\gamma = 0.4$, respectively. The value $\nu_R = 4.6$ is given by our calculated phase shifts for $|\lambda| = 0.37$ with an associated reduced width $\gamma = 0.5$. The calculated S-wave scattering lengths do not agree with the analysis of 3-pion τ decay,[8,9] but as this analysis is based on the assumption of small S-wave scattering lengths (i.e., α_0, $\alpha_2 \ll 1$), the calculated values are not necessarily in disagreement with experiment.

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