PROTON-PROTON INTERACTION*

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We have investigated the applicability of the modified boundary condition model[1,2] to the extensive proton-proton scattering data in the energy range extending up to 350 Mev. The success obtained with the model using a field-theoretical potential tail indicates at once the validity of the one- and two-pion exchange potentials (TPEP) in their appropriate range, and the physical significance of the energy-independent boundary condition employed to represent the shorter range interaction. The precision fit we have obtained requires only nine parameters plus the acceptance of the usual fourth-order static "perturbation" theory results for the potential tail.

For $r > r_0$ (r_0 is the same in all states), we assume a potential of the form

$$V = V_2 + V_4,$$

$$V_2(r) = f^2\mu c^2 \frac{\vec{\tau}^1\cdot\vec{\tau}^2}{3}\left[\vec{\sigma}^1\cdot\vec{\sigma}^2 + S_{12}\left(1+\frac{3}{\mu r}+\frac{3}{(\mu r)^2}\right)\right]\frac{e^{-\mu r}}{\mu r},$$

$$V_4(r) = f_1^4\mu c^2\left[-\left(\frac{2M\lambda}{\mu}\right)^2\frac{6}{\pi(\mu r)^2}K_1(2\mu r) + \left(\frac{2M\lambda}{\mu}\right)6\left(\frac{1+\mu r}{(\mu r)^2}\right)^2 e^{-2\mu r} - R_1(\mu r) + \vec{\sigma}^1\cdot\vec{\sigma}^2 R_2(\mu r) + S_{12}R_3(\mu r)\right], \tag{1}$$

where

$$R_1(x) = \frac{2}{\pi}\left\{\vec{\tau}^1\cdot\vec{\tau}^2\left[\frac{12}{x^2}+\frac{23}{x^4}K_1(2x)+\left(\frac{4}{x}+\frac{23}{x^3}\right)K_0(2x)\right]\right.$$
$$\left. + \xi(3-2\vec{\tau}^1\cdot\vec{\tau}^2)\left[\left(\frac{1}{x^2}+\frac{4}{x^3}+\frac{4}{x^4}\right)K_1(x)+\left(\frac{1}{x}+\frac{2}{x^2}+\frac{2}{x^3}\right)K_0(x)\right]e^{-x}\right\},$$

$$R_2(x) = \frac{2}{\pi}\left\{-\left[\left(\frac{8}{x^2}+\frac{12}{x^4}\right)K_1(2x)+\frac{12}{x^3}K_0(2x)\right] + \tfrac{2}{3}\xi(3-2\vec{\tau}^1\cdot\vec{\tau}^2)\left[\left(\frac{1}{x^2}+\frac{2}{x^3}+\frac{2}{x^4}\right)K_1(x)+\left(\frac{1}{x^2}+\frac{1}{x^3}\right)K_0(x)\right]e^{-x}\right\},$$

$$R_3(x) = \frac{2}{\pi}\left\{\left[\left(\frac{4}{x^2}+\frac{15}{x^4}\right)K_1(2x)+\frac{12}{x^3}K_0(2x)\right] - \tfrac{1}{3}\xi(3-2\vec{\tau}^1\cdot\vec{\tau}^2)\left[\left(\frac{1}{x^2}+\frac{5}{x^3}+\frac{5}{x^4}\right)K_1(x)+\left(\frac{1}{x^2}+\frac{1}{x^3}\right)K_0(x)\right]e^{-x}\right\}.$$

f and f_1 are the dimensionless rationalized pseudovector coupling constant, and are approximately $(0.08)^{1/2}$ when evaluated from meson scattering. We distinguish the numerical values of f and f_1 in order to test separately the significance of V_2 and V_4. μ and M are the pion and nucleon masses, respectively. A variable "pair suppression" is provided by the parameter λ. We shall call ξ the "ladder" parameter. The other operators and functions are standard. $\hbar = c = 1$.

For $f_1^2 = f^2$ and $\xi = 1$, V is the TMO potential.[3] For $\xi = 0$ we have the BW potential.[4] Thus the terms proportional to ξ represent the contribution of the iterated (ladder) second-order diagram to the fourth-order potential which has been the subject of so much controversy.[5] It should be noted that pair suppression ($\lambda < 1$) is a nonperturbative result (or at best comes from high-order terms) but is usually included in the perturbative potentials. No spin-orbit term has been included, as both the phenomenological evidence[6] and the theory[7] indicate a very short range behavior. On the basis of earlier results[1,2] the boundary radius r_0 is expected to include the spin-orbit region. The phase shifts and the corresponding experimental uncertainties, as listed in Table I, were taken from Breit et al.[8] and from effective-range parameters. Only 0, 95, 210, and 310 Mev were fitted in the expectation that the results for intermediate energies would interpolate well.

Starting with Coulomb functions at $r = 8$ f, the wave functions at each energy were numerically integrated inwards (on the IBM-709 at M.I.T.) to r_0, for a set of "potential parameters" r_0, f_1^2, λ, and ξ. For some runs f^2 was fixed at 0.08; for others we set $f^2 = f_1^2$ and they were varied together. At r_0, the logarithmic derivative of the wave function $F_{Jls} = r_0[(1/\psi_{Jls})d\psi_{Jls}/dr]_{r=r_0}$ was evaluated. For coupled states,[7] F is a 2×2 Hermitian matrix,

$$\begin{pmatrix} F_{J,J-1,1} & F_J^{\,c} \\ F_J^{\,c*} & F_{J,J+1,1} \end{pmatrix}.$$

At zero energy the scattering length, a_0, was used to determine the asymptotic $^1S_0$ wave function, which was then integrated inward to obtain F_{000}(0 Mev). The singlet effective range $r_{\rm eff}$(calc)

Table I. Model parameters at minimum.

$f^2 = f_1^2 = 0.0837$; $r_0 = 0.7011$ f; $\lambda = 1.00$; $\xi = 0.50$; $M = 11.10$; ($M_0 = 14$)

Jls	Energy (Mev)	δ (radians)[a]	$\Delta\delta$ (radians)	F	ΔF	$\bar{F}$
000	0	b		1.02	0.06	
	95	0.419	0.026	1.15	0.07	
	210	0.087	0.026	1.05	0.08	
	310	-0.175	0.044	1.12	0.14	1.08
220	95	0.068	0.012	1.63	4.43	
	210	0.140	0.017	3.66	2.39	
	310	0.192	0.017	3.25	0.94	3.14
011	95	0.209	0.044	-7.33	2.42	
	210	-0.035	0.061	-7.14	2.06	
	310	-0.209	0.035	-8.65	1.55	-7.82
111	95	-0.209	0.009	10.11	6.37	
	210	-0.364	0.017	17.34	12.36	
	310	-0.480	0.026	15.21	9.70	13.37
211	95	0.168	0.009	0.11	0.46	
	210	0.311	0.017	-0.78	0.35	
	310	0.291	0.017	-0.19	0.12	-0.26
$J=2$ coupling[c]	95	-0.279	0.070	-0.14	0.17	
	210	0.131	0.035	0.13	0.16	
	310	-0.105	0.035	0.03	0.14	0.01
231	95	-0.003	0.008	-4.04	0.05	
	210	0.003	0.026	-3.98	0.09	
	310	0.017	0.017	-4.43	0.27	-4.06
331	95	-0.026	0.052	-3.04	115.74	
	210	-0.039	0.017	-3.34	0.73	
	310	-0.062	0.017	4.31	55.29	-3.24

[a] Nuclear Blatt-Biedenharn phase shifts.
[b] $a_0 = -7.7 \pm 0.1$ f; $r_{\rm eff}$(exp) $= 2.67 \pm 0.03$ f; $r_{\rm eff}$(calc) $= 2.68$ f.
[c] For δ read the nuclear Blatt-Biedenharn coupling parameter ϵ_2.

was then calculated directly from F_{000} and the potential parameters.

The F_{Jls} so obtained were in general energy dependent and $r_{\text{eff}}(\text{calc})$ did not agree with $r_{\text{eff}}(\text{exp})$. The potential parameters were then varied to make the f's as energy-independent as possible and to bring $r_{\text{eff}}(\text{calc})$ into agreement with $r_{\text{eff}}(\text{exp})$. To do this in a statistically significant way, utilizing the experimental uncertainties in the phase shifts $\Delta\delta_{Jls}$ and in the effective-range parameters Δa_0 and $\Delta r_{\text{eff}}(\text{exp})$, the function

$$M=\sum_{JlsE}\left(\frac{F_{Jls}(E)-\bar{F}_{Jls}}{\Delta F_{Jls}(E)}\right)^2+\left(\frac{r_{\text{eff}}(\text{calc})-r_{\text{eff}}(\text{exp})}{\Delta r_{\text{eff}}(\text{exp})}\right)^2$$

was calculated and then minimized (using a gradient search) with respect to the potential parameters.

$$\bar{F}_{Jls}=[\textstyle\sum_E F_{Jls}(E)/\Delta F_{Jls}(E)]/\sum_E[\Delta F_{Jls}(E)]^{-1}$$

is the weighted average of the F_{Jls} at all different energies. $\Delta F_{Jls}(E)=(\partial F_{Jls}/\partial\delta_{Jls})\Delta\delta_{Jls}$, where the $\Delta\delta_{Jls}$ are the experimental uncertainties from Table I. Note that $(\partial F/\partial\delta)^{-1}$ measures the sensitivity of δ to F. Those δ's in the sum in (2) which are insensitive to F have a reduced relative importance.

The most probable value of M, corresponding to 50% correlation probability, M_0, is given by the number of phase shifts less the number of F_{Jls} less the number of potential parameters. Counting the relevant states and energies, as seen in Table I, $M_0=14$. $M_0<14$ implies that if one puts $F=\bar{F}$ at all energies, the corresponding phase shifts will usually lie within their experimental limits. The word "usually" is meant in its appropriate statistical sense.

Equation (2) neglects the effect of correlations between the phase-shift errors because they are difficult to take into account numerically. The effect of including the correlations in M is usually to decrease its value. Thus a "good fit" remains a good fit while a "bad fit" may become a "good fit" if the correlations are very strong. Thus our criterion suffices for an acceptable potential but can only weakly reject one. However, if $M\gg M_0$ an unusual set of correlations would be required to make the potential acceptable. We intend to inquire further into the effect of correlations.

With $f^2=0.08$, a minimum of $M=10.73$ was obtained for $f_1^2=0.0863$, $\lambda=1.000$, $\xi=0.500$, and $r_0=0.7011$ f. With $f^2=f_1^2$, a minimum of $M=11.10$ was obtained for $f^2=f_1^2=0.0837$ and no change in the other parameters. The values of the $\bar{F}_{Jls}$, F_{Jls}, ΔF_{Jls}, and $r_{\text{eff}}(\text{calc})$ corresponding to the latter case are given in Table I. The values for r_0 and $\bar{F}_{000}$ are very similar to those obtained previously for special forms of the modified boundary condition model.[2]

It is highly significant that the best fit is obtained for the coupling constant determined by pion-nucleon data,[9] $f^2\approx f_1^2\approx 0.082$. A variation of $f_1^2=f^2$ of about 0.002 would increase M by unity. Previous OPEP analysis[10] had determined that $f^2\simeq 0.08$ was required for the higher angular momenta. We see at once a similar result for the TPEP region and for lower angular momenta, but not independently of the model for the internal region. However, if this model was not a significant description of the scattering, it would be expected that $M\gg M_0$. The parameters λ and ξ fall within the expected range. The sensitivity to λ and ξ is small however. Varying ξ between 0 and 1 only increases M to 12. This is presumably due in part to the negligible difference between TMO and BW in the singlet even state. Putting $\lambda=0$ while the other potential parameters remain unchanged makes $M=58$. If we then minimize holding $\lambda=0$, we obtain $M=24.8$, but require $f^2=f_1^2=0.093$. This complete pair suppression is not consistent with our model, especially if we require $f^2\approx f_1^2\approx 0.08$.

There is then a strong statistical indication of the quantitative correctness of the fourth-order p-p static potential within limits of the order of the size of the "ladder" contribution. Conversely the energy-independent boundary condition is a statistically satisfactory representation of the inner region. If the fourth-order perturbation static potential is accepted on theoretical grounds for $r>r_0$, then the parameters given by the $\bar{F}$'s and r_0 are sufficient to account for all the scattering data. This is a much smaller number of parameters than those inherent in previous proposed phenomenological potentials.[6] The fact that $r_0=0.7$ f indicates that the TPEP approximation breaks down at this distance and implies the onset of nonlocal or energy-dependent effects. In more conventional representations the short-range spin-orbit potential has the same implication.

For $l>3$ the phase shifts are independent of

F_{Jls} except for a very narrow range which gives rise to a sharp resonance.[11] Thus these phase shifts can in general be considered to depend only on TPEP. For example, a test of TPEP independent of the boundary condition can be made by examining the ΔF's for high angular momenta at low energy.[12] The extremely large value of ΔF_{331} $\times$(95 Mev) is such a verification of TPEP, and the decrease of ΔF_{220} with increasing energy is also significant.

The following calculations are now in progress: (1) The phase shifts for higher l are being evaluated with our potential to further test TPEP. (2) The phase shifts at interpolating energies are being calculated. (3) The nonperturbative potentials KMO[13] and Klein-McCormick[14] are also being tried in place of Eq. (1). (4) The neutron-proton phase-shift data[15] are under analysis employing the modified boundary condition model with the potential forms used in the p-p analysis. Here the triplet even potential depends strongly on the ladder ambiguity so that one may hope to obtain a better value of ξ.

The above extensions of the analysis will be published elsewhere together with a more detailed discussion of the method and results of this paper. We conclude from the present analysis that the field-theoretic fourth-order static potential combined with an energy-independent boundary condition is a simple and accurate, physically significant representation of the data.

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