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[7] See, for example, N. F. Ramsey, Molecular Beams (Oxford University Press, New York, 1956), p. 119; or W. E. Lamb, Jr., Lectures in Theoretical Physics (Interscience Publishers, New York, 1960), Vol. II, p. 444.

[8] K. Shimoda, T. C. Wang, and C. H. Townes, Phys. Rev. 102, 1308 (1956).

[9] See, for example, J. R. Singer, Masers (John Wiley & Sons, New York, 1959).

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[11] E. L. Hahn, Phys. Rev. 77, 297 (1950).

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# POLARIZATION AND THE TRIPLET ELECTRON-HYDROGEN SCATTERING LENGTH

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Recently, upper bounds for the triplet and singlet scattering lengths of electrons scattered from atomic hydrogen have been calculated by Rosenberg, Spruch, and O'Malley.[1] In the triplet case the upper bound was found to be $a_t \leq 1.91$.[2] This bound was obtained from an expression involving an arbitrary wave function $\psi$ having a correct asymptotic form of an $s$-scattered wave at infinity. The particular function used by Rosenberg et al. did not contain a slowly vanishing part ($\sim -r^{-2}$) coming from the induced polarization in the hydrogen atom. A recent calculation[3] in which such a term was included gave a scattering length practically identical with the RSO bound, from which it may have been inferred that the polarization has an almost negligible effect on the scattering length and that the value $a_t = 1.91$ is very close to the correct scattering length. We wish to show that the polarization has a very important effect on the scattering length (of the order of 10%). Secondly, we want to report an accurate calculation of the triplet scattering length:

$$a_t = 1.74, \tag{1}$$

where the error is estimated to be less than three percent. The difference between this value and that of RSO is almost wholly due to the long-range polarization.

The effect of polarization as well as the calculation may be understood from a method which has been outlined in this journal.[4] The basic formula of that approach is

$$\sin(\delta-\delta_0) = \frac{-1}{k}\sum_{l=1}^{\infty}\frac{2}{(2l+1)^{1/2}}\int_0^{\infty} dr_1 \int_0^{r_1} dr_2 \Phi_0^{(0)} \frac{r_2^{\,l}}{r_1^{\,l+1}} \Phi_l. \tag{I.5}$$

$\delta$ is the exact phase shift of the whole problem and $\Phi_l(r_1 r_2)$ are "coefficients" in the expansion of the exact $s$-wave wave function in terms of Legendre polynomials $P(\cos\theta_{12})$, Eq. (I.1). $\delta_0$ is the phase shift associated with a zeroth order problem, Eq. (I.4), which in particular does not contain any induced multipole distortion (polarization) effects.

Assume that we have computed $\delta_0$ and all $l > 1$ terms on the right-hand side of (I.5). Assume also that only that part of the $l = 1$ term has been calculated by integrating up to a point $R$ beyond which the difference between $\Phi_1$ and its asymptotic form,

$$\lim_{r_1\to\infty} \Phi_1 = \frac{-2}{\sqrt{3}}\frac{\sin(kr_1+\delta)}{r_1^{\,2}} e^{-r_2}\left(\tfrac{1}{2}r_2^{\,3} + r_2^{\,2}\right), \tag{I.6}$$

can be considered negligible. Let $\delta(R)$ be the phase shift so calculated, and let $a(R)$ be the associated scattering length:

$$\lim_{k\to 0} \delta(R) = \pi - a(R)k.$$

$a \equiv a(\infty)$ is the exact scattering length. We find from the various asymptotic forms

$$\frac{1}{k}[\delta-\delta(R)] \cong \frac{1}{k^2}\frac{8}{3}\int_R^\infty dr_1 \int_0^{r_1} dr_2 \sin^2(kr_1+\delta)e^{-2r_2}(\tfrac{1}{2}r_2^4 + r_2^3)$$

$$\cong \frac{\alpha}{k^2}\int_R^\infty \frac{\sin^2(kr_1+\delta)}{r_1^4}dr_1, \tag{2}$$

where $\alpha = 9/2$ is the polarizability of hydrogen. The right-hand side of (2) can be simply carried out to order $k$:

$$\frac{1}{k}[\delta-\delta(R)] = \left(\frac{\alpha}{3}\frac{a^2}{R^3} - \alpha\frac{a}{R^2} + \alpha\frac{1}{R}\right) + k\left(\frac{\pi\alpha}{3}\right) + O(k^2 \ln k). \tag{3}$$

The linear term in $k$ induces a linear term in the effective-range expansion of $k \cot\delta$:

$$k \cot\delta = -\frac{1}{a} + \left(\frac{\pi\alpha}{3a^2}\right)k + \cdots, \tag{4}$$

and the coefficient of the $k$ term is precisely that derived by Spruch et al.[5] for the idealized problem of a particle scattered from a $-\alpha/r^4$ potential.

However, of prime significance to us is the constant term in (3), i.e., the formula

$$a = a(R) - \alpha\left(\frac{1}{3}\frac{a^2}{R^3} - \frac{a}{R^2} + \frac{1}{R}\right). \tag{5}$$

Clearly, a calculation which includes no long polarization gives rise to an approximation to $a(R)$, which will be substantially greater than the true scattering length, $a$.

Our own calculations were carried out to $R = 20$. We obtain $a(R) = 1.944$. Now applying (5) gives the final value $a = 1.74$.

The same considerations also apply to the singlet scattering; however, since $a \approx 6$ is much larger in that case, the decrease in $a(R)$ will be smaller.

We expect to have accurate phase shifts for both singlet and triplet scattering shortly. These together with details of the method will be published at that time.

I should like to thank Miss Dorothy Hoover for programming the $\delta_0$ calculation. I am greatly indebted to Dr. L. Spruch and Dr. T. F. O'Malley whose suggestion that we look at the long-range potential effect on the phase shift has led directly to this Letter. The helpful correspondence of Dr. Charles Schwartz is also acknowledged.

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[1] L. Rosenberg, L. Spruch, and T. F. O'Malley, Phys. Rev. 119, 164 (1960). We refer to this as RSO.

[2] Our units are lengths in Bohr radii, energy in rydbergs.

[3] A. Temkin and J. C. Lamkin, Phys. Rev. 121, 788 (1961). Integrations of the differential equations were carried out to $r = 20$. The considerations of this Letter indicate that if we had carried out the integration to large enough values of $r$, we would have obtained substantially the same result as Eq. (1).

[4] A. Temkin, Phys. Rev. Letters 4, 566 (1960). Equations referring to this article will be prefixed by I.

[5] L. Spruch, T. F. O'Malley, and L. Rosenberg, Phys. Rev. Letters 5, 375 (1960).