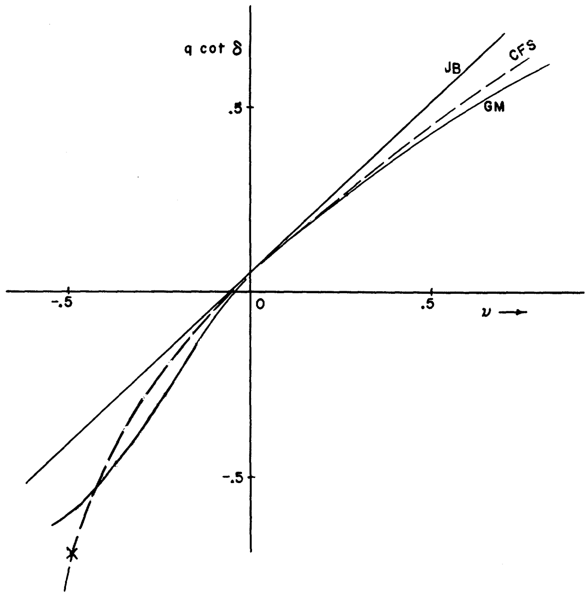


FIG. 2. Results of the present calculation (GM), compared to those of Cini et al.[5] (CFS). The straight line (JB) is that of Jackson and Blatt.[4]

circle so the approximation is not expected to be valid. However, by removing the restriction that the pole be at the center of the circle, one could exactly fit their point at $\nu = -\frac{1}{2}$ and still have a two-parameter fit. Note that by introducing higher multipoles one could develop a systematic approximation scheme.

The shape parameter has the same sign as that of Cini et al.,[5] and also as that of Noyes and Wong,[7] which may well be a consequence of the values of the effective range and scattering length and the location of the singularities in the Mandelstam representation, even though an experimentally meaningful shape dependence might be critically affected by relativistic effects, Coulomb effects, and many other diverse effects.[8] One further advantage of this solution is that it preserves the left-hand branch line, which the pole approximation does not, and $h(\nu)$ is real only between $-\frac{1}{4} < \nu < 0$.

Frazer[9] has developed a conformal mapping method for use in the Mandelstam representation, though we are unfamiliar with the details of this work.

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# SINGULARITIES OF THE COSMOLOGICAL SOLUTIONS OF GRAVITATIONAL EQUATIONS

I. M. Khalatnikov, E. M. Lifshitz, and V. V. Sudakov
The Institute of Physical Problems, Academy of Sciences, Moscow, U.S.S.R.
(Received February 28, 1961)

In cosmological applications of the general relativity theory extensive use is made of the well-known (Friedmann's) solution of the Einstein gravitational equation which is based on the assumption of complete homogeneity and isotropy of the space distribution of matter. This assumption is far reaching in its mathematical aspects, not to mention that its fulfillment in the actual universe could at best be only of approximate nature. Hence the question arises: To what extent does the important property of the resulting solution—the existence of the time singularity, depend on this specific assumption? The solution of this problem, which is of primary importance for the entire cosmology, requires an investigation of the situation arising for a quite arbitrary distribution of matter and gravitational field in space. A short summary is given here of the results of such an investigation.

The natural choice of the reference system in dealing with this problem turns out to be a system, subject to the conditions $-g_{00} = 1$, $g_{0\alpha} = 0$, $\alpha = 1, 2, 3$ (we shall call such a system synchronous, since it allows of the synchronization of clocks along the entire space). It was long ago pointed out by Landau that due to one of the gravitational equations (the 00 equation) the metric determinant $g$ must inevitably become zero in a finite time. However, this result (which was recently found independently also by other authors[1]) does by no means prove—contrary to the opinion expressed in the literature—the inevitability of the existence of a real (physical) singularity in the metric, which cannot be excluded by any transformation of the reference system. The singularity can turn out to be fictitious, nonphysical, being connected merely with the specific nature of the chosen reference system.

An answer to this question emerges from the geometrical analysis of the space-time properties in the synchronous system of reference.

It is easily seen that in a synchronous reference system the lines of time are geodesics in the 4-space. This property can be used for a geometrical construction of such a system in any space-time. We choose an arbitrary space-like hypersurface and construct a set of geodesics normal to this hypersurface. If one defines now the time coordinate as the length of a geodesic between a given world point and the intersection with the hypersurface, one arrives, as it is easy to see, at a synchronous reference system.

But geodesic lines of an arbitrary set in general intersect each other on some envelope hypersurfaces—the four-dimensional analogs of the caustic surfaces of geometrical optics. Thus there exists a geometrical reason for the appearance of a singularity, which is due to specific properties of the synchronous reference system and is therefore obviously of a nonphysical nature. It is to be emphasized, however, that an arbitrary metric of a 4-space in general allows also for the existence of nonintersecting sets of time-like geodesics. But the above-mentioned property of the gravitational equations means that the metric admitted by them excludes the possibility of the existence of such sets, so that the lines of time necessarily intersect each other in any synchronous reference system.

This means, from the analytical point of view, that in a synchronous system of reference the Einstein equations have a general solution with a fictitious singularity with respect to time.

Thus any foundation is removed for the existence, along with this general solution, of yet another, which would also be a general one but would have a real singularity. The criterion of the generality of the solution is the number of arbitrary functions (of the space coordinates) it contains. Among these functions there are in general also such, whose arbitrariness is due merely to the freedom in the choice of the reference system admitted by the equations. What is essential is only the number of the "physically different" arbitrary functions, which cannot be decreased by any specific choice of the reference system. For the general solution this number must be eight; these functions must provide for the possibility to put arbitrary initial conditions, determining the initial space distributions of the density and the three velocity components of the matter, and of the four quantities which determine the free gravitational field. (The latter number can be arrived at, e.g., by considering weak gravitational waves; since these waves are transverse, their field is characterized by two quantities which obey differential equations of the second order, and therefore the initial conditions for this field must be given by four space functions.)

The above geometrical considerations do not exclude, of course, the possibility of the existence of narrower classes of cosmological solutions with a real singularity. Indeed an extensive search (carried out by two of us[2]) for such solutions has shown that the widest of them contain only seven physically different arbitrary functions, i.e., one less than it is required for a general solution; hence even this solution in spite of its wideness is only a special case. In other words, this solution is unstable; there exist small perturbations which lead to its dissipation. Since in the synchronous reference system the singularity cannot disappear entirely, this means that it must go over, as a result of the perturbation, into a fictitious one.

Thus we are led to the fundamental conclusion that the existence of a physical time singularity is not an obligatory property of the cosmological models of the general relativity theory. The general case of an arbitrary distribution of matter and gravitational field leads to an absence of such a singularity.

This result is formally equally valid for the singularities towards both directions of time. However, physically these directions are of course not equivalent and there is an essential difference between both cases already in the statement of the problem itself. The singularity in the future can have a physical meaning only if it is admitted by quite arbitrary conditions given at any previous moment of time. On the other hand, it is perfectly clear that there are no reasons at all for the distribution of matter and field attained in the course of the evolution of the universe to comply with the specific conditions which are necessary for realization of the special solution with a physical singularity. Even if one admits the realization of such a specific distribution at some moment of time, it will inevitably be violated in the following time already as a result of the unavoidable fluctuations. Therefore the above results exclude the possibility of the existence of a singularity in the future; this means that the contraction of the universe (if it is at all to come) must afterwards change again to an expansion. As to the singularity in the past, an investigation based only on the gravitational equations, can only impose certain restrictions on the admissible character of the initial conditions, the complete elucidation of which is impossible in the framework of the existing theory.

A detailed account of this work will be published in the Journal of Experimental and Theoretical Physics (U.S.S.R.).

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## DISPERSION RELATIONS FOR PRODUCTION AMPLITUDES

Y. S. Kim
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
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Dispersion relations for production amplitudes were first proposed by Polkinghorne.[1] His work was extended later by Kibble.[2] Logunov and other authors[3-5] have published a series of papers in which they obtain single-variable dispersion relations for various production processes. They presented a rigorous proof for the double Compton effect.[5] On the other hand, for the process of pion production in pion-nucleon collision, their discussion is based on a rather delicate theorem on analyticity in the pion mass. In this note we shall demonstrate, by means of a concrete example from perturbation theory, that the conjectured dispersion relation is in fact not valid, i.e., that the production amplitude has complex singularities when regarded as a function of the Logunov variable.

For the process $\pi+N\rightarrow\pi+\pi+N$, we take the five independent variables in the form:

$$\begin{aligned}E&=-k\cdot(p+p'),\\ \eta&=(k'-k'')\cdot(p+p')/(k'+k'')\cdot(p+p'),\\ x_1&=-k'\cdot(p-p'),\\ x_2&=-k''\cdot(p-p'),\\ v&=(p-p')^2,\end{aligned}\tag{1}$$

where $p$ and $p'$, respectively, denote the four-momenta of the incoming and outgoing nucleons; $k$ refers to the incoming pion; $k'$ and $k''$ refer to the two outgoing pions.

$$p^2=p^2=-m^2,\quad k^2=k'^2=k''^2=-\mu^2;\tag{2}$$

$m$ and $\mu$ represent, respectively, nucleonic and pionic masses.

Logunov and his co-workers define the following two four-vectors in the Breit system where $\vec{p}+\vec{p}'=0$:[4]

$$\begin{aligned}A&=\tfrac{1}{2}\left[\alpha^{-1/2}(1-\xi)k'+\alpha^{1/2}(1+\xi)k''\right],\\ B&=\tfrac{1}{2}\left[\alpha^{-1/2}k'-\alpha^{1/2}k''\right],\end{aligned}\tag{3}$$

where

$$\alpha=k_0'/k_0'',\quad \xi=-\mu^2(\alpha^{-1}-\alpha)/(2B^2),$$

and choose $A_0$, $A^2$, $\vec{B}\cdot\vec{p}$, $\alpha$, and $\vec{p}^2$ as the five independent variables. Then they investigate analyticity in $A_0$ and assert a cut-plane representation for the pion production amplitude for fixed physical values of the other four variables. We shall show in the following that a simple analysis in perturbation theory leads to the conclusion that the existence of complex singularities is inevitable.