NONCLASSICAL STRUCTURE OF THE ENERGY-MOMENTUM TENSOR OF A POINT MASS SOURCE FOR THE SCHWARZSCHILD FIELD*

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Although the Schwarzschild solution to the field equations has been known for many years, because of the complicated nonlinear character of the equations, there is still no generally accepted form for the structure of the energy-momentum tensor $T_\nu{}^\mu$ when the latter is that of a point mass source. From a classical standpoint, one would surmise that $T_\nu{}^\mu$ in a static coordinate system reduces to $T_0{}^0$, with $T_0{}^0$ proportional to a delta function, but from the structure of the field equations it is by no means obvious that other components are not also present.

Indeed, as we shall now show, and briefly discuss, this tensor does not have only one nonvanishing component, i.e., $T_0{}^0$ (as one might expect classically for a point particle at rest), but four: $T_0{}^0=T_1{}^1$, $T_2{}^2=T_3{}^3=-\frac{1}{2}T_0{}^0$, with $T_0{}^0=m\delta(r)/2\pi r^2$; the notation is described below.

In order to arrive at this result, we shall first show that, for systems with static spherical symmetry, when a certain condition is satisfied by the $g_{\mu\nu}$ (as is the case for the Schwarzschild solution), the field equations reduce to a pair of dependent linear equations, most simply expressed in terms of the scalar potential U (defined by $g_{00}=1+2U$). Using these equations, the above structure for $T_\nu{}^\mu$ may be immediately inferred.

Upon introducing the coordinate system associated with the following standard line element for systems with static spherical symmetry,

$$ds^2=g_{00}(r)\,dt^2+g_{11}(r)\,dr^2-r^2(d\theta^2+\sin^2\theta\,d\phi^2), \quad (1)$$

one finds for the nonvanishing components of the field equations[1,2]:

$$G_0{}^0=-r^{-2}[1+(rg^{11})_{,r}]=-\kappa T_0{}^0,$$

$$G_1{}^1=G_0{}^0+r^{-1}(g^{11}{}_{,r}-g^{00}g^{11}g_{00,r})=-\kappa T_1{}^1,$$

$$G_2{}^2=-g^{00}g^{11}(\tfrac{1}{2}\nabla^2 g_{00})-[\tfrac{1}{2}+\tfrac{1}{4}r(\ln g_{00})_{,r}](G_1{}^1-G_0{}^0)$$
$$=-\kappa T_2{}^2, \quad (2)$$

and by spherical symmetry, $G_3{}^3=G_2{}^2$. We note the following about the equations: (a) The first equation is already linear in g^{11}; (b) the other two equations would be linear if $g^{00}g^{11}$ were a constant,[3] since this is equivalent to $g^{11}{}_{,r}-g^{00}g^{11}g_{00,r}=0$; by the asymptotic condition[4] this constant may be set equal to -1; (c) if $g^{00}g^{11}=-1$, the transverse equation reduces to a Newtonian one, relating $\nabla^2 g_{00}$ to the transverse stresses; (d) in order for $g^{00}g^{11}$ to be constant, it is necessary and sufficient that $T_0{}^0=T_1{}^1$. When the latter is the case, the requirement that $T_\nu{}^\mu{}_{;\mu}=0$ reduces to

$$(r^2T_1{}^1)_{,r}-2rT_2{}^2=0, \quad (3)$$

and it follows that there is only one independent component for the $T_\nu{}^\mu$ which most conveniently may be replaced by the trace, T. One then has in terms of T for the components

$$T_0{}^0=T_1{}^1=r^{-4}\int Tr^3\,dr+C/r^4,$$

$$T_2{}^2=T_3{}^3=\tfrac{1}{2}T-T_0{}^0, \quad (4)$$

where C is a constant of integration.[5]

Let us now consider the Schwarzschild field, i.e., $g_{00}=1-2Gm/r$, $g^{11}=-g_{00}$. This field is derived under the assumption that outside the source $T_\nu{}^\mu=0$; hence in this region, the requirements on $T_\nu{}^\mu$ for linearity are satisfied. We shall continue the solution down to the source point at $r=0$, noting that $g_{00}(=-g^{11})$ is finite and differentiable even when passing through the Schwarzschild singularity, and the regularity of the field equations is not altered by the fact that $g_{00}=0$ (or $g_{11}=\infty$) at $r=2Gm$, since this is not a singularity in the differential equations themselves.[6]

Upon introducing the scalar potential, $-g^{11}=g_{00}=1+2U$, the field equations reduce to the following pair of dependent linear equations[7]:

$$-2r^{-2}(rU)_{,r}=\kappa T_0{}^0(=\kappa T_1{}^1),$$

$$-\nabla^2U=\kappa T_2{}^2(=\kappa T_3{}^3), \quad (5)$$

where, for the case of a point particle, the source terms are singular distributions which vanish for $r>0$. For the Schwarzschild solution, i.e., $U=-Gm/r$, one readily obtains the following values for the $T_\nu{}^\mu$ which may be conveniently summarized as

$$T_\nu{}^\mu = \mathrm{diag}\ (1,\ 1,\ -\tfrac{1}{2},\ -\tfrac{1}{2})\, m\delta(r)/2\pi r^2, \qquad (6)$$

where $\delta(r)$ is the radial delta function $\int_0^\infty \delta(r)\,dr = \frac{1}{2}$. Since $r\delta'(r) = -\delta(r)$, these distributions satisfy the requirement (3).

Although the m appearing in the above equations is up to this point of the discussion to be thought of only as a gravitational mass, it follows directly from the action principle that it is also a dynamical quantity, the rest mass. We have for the gravitational action, A, the expression $\int(-g)^{1/2}\kappa^{-1}R\,d^4x$ which by the field equations may be written $\int(-g)^{1/2}T\,d^4x$. Hence, since $T = T_0{}^0 = m\delta(r)/2\pi r^2$, the action integrated for a finite time interval over any spatial region ω containing the particle is

$$A(t;\ \omega) = \int_{t_0}^{t} dt \int_\omega \frac{m\delta(r)}{2\pi r^2}\, d^3r = m(t - t_0). \qquad (7)$$

The surfaces of constant action are therefore parallel to the surfaces of constant time, and for this case, and in general for static coordinate systems,[8] the normal derivative to the action surface is the rest mass, i.e., $dA/dt = m = \int(-g)^{1/2}T\,d^3x$.[9]

The result is therefore quite analogous to special relativity because in the above case $T_1{}^1 + T_2{}^2 + T_3{}^3 = 0$, so that $T_0{}^0 = T$ as in special relativity for a particle at rest. However, despite this analogy, we encounter a difficulty if we try to describe $T_\nu{}^\mu$ classically. We would expect $T_\nu{}^\mu = \rho(dx^\mu/ds)(dx_\nu/ds)$, where ρ is the trace or proper density $m\delta(r)/2\pi r^2$. However, the previously found values for $T_\nu{}^\mu$ would imply, formally, in the limit $r = 0$,

$$g_{00}(dt/ds)^2 = 1, \quad g_{11}(dr/ds)^2 = 1, \quad r^2(d\theta/ds)^2 = \tfrac{1}{2},$$
$$r^2\sin^2\theta\,(d\phi/ds)^2 = \tfrac{1}{2}, \qquad (8)$$

instead of as in special relativity $(dt/ds)^2 = 1$, with the other terms vanishing. Moreover, again formally, we have (since $|g_{00}| = \infty$, $g_{11} = 0$, at $r = 0$) $dt/ds = 0$, $|dr/ds| = \infty$, in contrast with the classical result $dt/ds = 1$, $dr/ds = 0$. The two angular terms may be formally equated to yield $d\theta^2 = \sin^2\theta\,d\phi^2$, which has the particular integral, $\sin\theta\,\exp|\phi| = \text{constant}$, corresponding to spirals on the unit sphere.[10]

While these results follow because we are no longer working in the classical region of space-time, and because of the structure of $T_\nu{}^\mu$, it would appear that we have reached a fundamental limitation here in understanding the point particle in general relativity as a purely classical object (i.e., a structureless mass point, with the spatial stresses vanishing).[11] We leave as an open question whether this nonclassical structure has some relation to the nonclassical behavior of particles as exhibited in the quantum domain.

In conclusion, to clarify the physical principle underlying the condition on the $T_\nu{}^\mu$, we note that one may arrive at this condition in the following way: Let a particle be freely falling radially in a gravitational field described by the line element (1); what is the condition on g_{00} and g_{11} that will make the radial acceleration independent of the energy integral $E = g_{00}\,dt/ds$? One readily finds that the condition is $g_{00}g_{11} = \text{constant}$, and upon examining the field equations, one finds $T_0{}^0 = T_1{}^1$. In other words, we are dealing here with a consequence of the principle of equivalence as it manifests itself in the field equations.[12] It is certainly remarkable that this condition should also make the equations, for this case, linear and, upon transforming to rectangular coordinates, make the coordinate system "proper" (to within a normalization constant). We hope to discuss these results further in a subsequent publication.

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[1]Because of the final linearity in g_{00} and g^{11}, it will not be convenient to make the customary substitution $g_{00} = \exp\mu(r)$, $g_{11} = -\exp\nu(r)$. However, if one adopts static cylindrical coordinates, one obtains an equation linear in $\ln g_{00}$, as first shown by H. Weyl, Ann. Physik 54, 117 (1917); 59, 185 (1919). See also T. Levi-Civita, Rend. Accad. nazl. Lincei, (1917-1919); R. Bach and H. Weyl, Math. Zeit. 13, 134 (1922). We should like to emphasize the restricted character of our results, i.e., to systems with static spherical symmetry, and to call attention to the more general problem of finding rigorous linearizing conditions for the field equations, and determining, by experiment, whether or not they do occur.

[2]We set $c = 1$, $\kappa = 8\pi G$, where G is the Newtonian gravitational constant, also $f_{,r} \equiv df/dr$, $\nabla^2 f \equiv r^{-2}(r^2 f_{,r})_{,r}$.

[3]This condition is not, however, necessary for linearity, e.g., $g_{00} = \text{constant}$ yields equations linear in g^{11}, although it, of course, requires a different structure for the $T_\nu{}^\mu$ than the Schwarzschild solution.

[4]That is, as $r \to \infty$, $g_{\mu\nu} \to \eta_{\mu\nu} = \mathrm{diag}\ (1, -1, -1, -1)$, to the extent that we may neglect cosmological effects.

[5]The term C/r^4, with $C = e^2/2$, where e is the charge of the electron in Heaviside units, occurs in the Reissner-Nordstrom solution, for which the $T_\nu{}^\mu$ also

satisfy the above linearity condition, as does also the cosmological term $\Lambda\delta_\nu{}^\mu$.

[6]Moreover, we note that the Schwarzschild solution for g_{00} is linear in m; hence we could alternatively reflect m and obtain a solution for which $g_{00} \geqslant 1$ for $0 < r \leqslant \infty$. One should therefore draw a clear distinction between singular points or regions in the field equations, and singularities or "unphysical" behavior in $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ due to the values of the $g_{\mu\nu}$.

[7]Note that upon substituting the expressions for $T_\nu{}^\mu$ in (5) into the requirement (3), it is satisfied identically, so that a reduced version of the Bianchi identities has been preserved.

[8]If one adopts the isotropic coordinates by setting $r = (1 + Gm/2\bar{r})^2\bar{r}$, one finds the interesting result that the particle cannot be located at $\bar{r} = 0$ in the isotropic system, and that rather $\bar{r} = -Gm/2$ corresponds to the particle's location. Moreover, the isotropic system for the range $0 \leqslant \bar{r} \leqslant \infty$ covers the space outside the Schwarzschild radius twice, so that $\bar{r} = 0$ actually corresponds to spatial infinity, as may be seen from the above transformation or by observing that the isotropic line element is form-invariant under the substitution $q = G^2m^2/4\bar{r}$. Compare the results, different from ours, obtained by R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. Letters 4, 375 (1960), who assume the particle to be located at $\bar{r} = 0$.

[9]We have also calculated the surfaces of constant action for Lemaître's nonstatic coordinate system [G. E. Lemaître, Ann. soc. sci. Bruxelles, Ser. A. 53, 51 (1933)], and find $A = m(t' + r')/2$. The original static observers satisfy $t' - r' =$ constant. Differentiating A in the direction $dt' = dr'$, we have again m.

[10]An alternative classical representation for the $T_\nu{}^\mu$ is given by $T_\nu{}^\mu = (\rho + p)(dx^\mu/ds)(dx_\nu/ds) - p\delta_\nu{}^\mu$, where p is a scalar "pressure." One then has $p = \rho/2$ and again, formally, $g_{00}(dt/ds)^2 = g_{11}(dr/ds)^2 = 1$, with the angular "motion" vanishing.

[11]One is perhaps reminded here of Einstein's feeling that one should not assume a priori a classical representation for the singularities of the gravitational field in connection with deriving the equations of motion from the field equations. See the interesting accounts in L. Infeld, Helv. Phys. Acta. Suppl. IV, 206 (1956); Revs. Modern Phys. 29, 398 (1957). The above results bring out the urgency of finding exact solutions to the two-body problem.

[12]The question of the relationship of this condition to the principle of equivalence has been raised recently by H. Bondi and S. Kilmister, Am. J. Phys. 28, 508 (1960), in their Letter commenting on the recent application of the principle by L. I. Schiff, Am. J. Phys. 28, 340 (1960). Note, however, that Schwarzschild's interior solutions do not in general satisfy this condition, in contrast with the interior solutions that would be obtained from (5).