Self-Consistent Equations for Variable-Velocity Three-Dimensional Inverse Scattering

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This paper considers the three-dimensional inverse scattering problem for the wave equation with variable velocity. A possible solution is presented in terms of equations whose self-consistent solution determines the velocity from scattering data. These self-consistent equations are (1) the wave equation in integral form, (2) a linear integral equation which relates the wave field and scattering data, and (3) a novel formula for the velocity in terms of the wave field.

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The inverse scattering problem is to recover information about an inaccessible region of space from measurements of scattered wave fields.[1,2] Consequently, inverse scattering theory is of fundamental interest for the basic and applied sciences. Important applications[3] range from radar and nondestructive evaluation to medical imaging and seismic exploration.

Well-understood exact inversion methods have been developed for problems whose parameters depend only on a single spatial dimension.[4] Such methods exist in a unified form and are widely useful in acoustics, electrodynamics, and elastodynamics, as well as quantum scattering. Furthermore, for these problems, accurate numerical algorithms are available.

The situation is much more difficult if the scatterer's properties depend on more than one spatial dimension. For the multidimensional case, the exact formulation of inversion methods for variable-velocity wave equations remains open. There are some techniques which are numerically tractable. However, these techniques are either approximate (e.g., optics, physical optics methods,[5] or the Born approximation[6]) or depend on brute-force variation of model potentials. In the latter method, the choice of a suitable variational potential requires substantial *a priori* information which is generally not available.

In this Letter we propose a general method, based on *linear* integral and differential equations, for solving the three-dimensional variable-velocity inverse scattering problem. The basis of the proposed method is a set of three equations. Their self-consistent solution (when it can be found) determines the velocity from the scattering data. These data are taken to be the scattering amplitude for *one direction* of incidence, all directions of scatter, and all frequencies. For ease of discourse, we restrict the velocity in the scattering region to be less than that of the embedding space. A similar, but considerably more complicated, set of equations can be found in the general case.

This Letter proceeds as follows. First, the three self-consistent equations will be given. Included is a brief derivation of a key equation [Eq. (5)] which relates the potential to the wave field. This derivation is based on low-frequency asymptotic behavior. Two of the equations are then combined to give a second equation [Eq. (11)] for the potential. Finally, some preliminary comments are made concerning the numerical solution of the self-consistent equations.

Our approach contains two interesting features. The first is that the key equation for the potential, (5), is based on low-frequency asymptotic behavior of the wave field. This is in sharp contrast to other approaches which consider either the high-frequency or short-time asymptotic behavior. The second interesting feature is that our formula (11) for the potential is similar to the trace formula introduced by Deift and Trubowitz for the one-dimensional case.[7]

We start with the variable-velocity wave equation

$$[\Delta + c^{-2}(\mathbf{x})k^2]\psi(k,\mathbf{x}) = 0. \tag{1}$$

Here Δ denotes the Laplacian, k is the magnitude of the wave vector, ψ describes a scalar field, and $\mathbf{x} \in R^3$ denotes the spatial coordinates. The velocity $c(\mathbf{x})$ is assumed to differ from 1 only in a bounded region situated about the origin of coordinates. Further, it is assumed that $c(\mathbf{x})$ is positive, bounded, and everywhere less than or equal to 1 [$c(\mathbf{x}) \le 1$]. This assumption will be needed for arguments involving causality.

We note that Eq. (1) can be derived from the wave equation that governs the propagation of sound in fluids (one sets the density equal to a constant). Consequently, it is suitable for modeling many problems in acoustics where the velocity is essentially independent of frequency. It is not as useful for modeling problems in electromagnetic scattering since in this case the velocity commonly varies with frequency.

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For scattering problems, Eq. (1) plus boundary conditions can be conveniently rewritten in integral form as[2]

$$\psi^{\pm}(k,\hat{\mathbf{e}},\mathbf{x}) = \exp(ik\hat{\mathbf{e}}\cdot\mathbf{x}) + k^2 \int d^3x' G_0^{\pm}(k,|\mathbf{x}-\mathbf{x}'|)V(\mathbf{x}')\psi^{\pm}(k,\hat{\mathbf{e}},\mathbf{x}'). \tag{2}$$

Here $V(\mathbf{x}) \equiv 1 c^{-2}(\mathbf{x})$, $\hat{\mathbf{e}}$ is a unit vector denoting the direction of incidence, and $G_0^{\pm}$ are given by

$$G_0^{\pm}(k,|\mathbf{x}-\mathbf{x}'|) = -(4\pi|\mathbf{x}-\mathbf{x}'|)^{-1}\exp(\pm ik|\mathbf{x}-\mathbf{x}'|). \tag{3}$$

The plus sign corresponds to outgoing (radiation) boundary conditions, while the minus sign corresponds to incoming boundary conditions. Finally, we define $\psi^{\mathrm{sc}} = \psi - \exp(ik\hat{\mathbf{e}}\cdot\mathbf{x})$.

An essential feature of our approach is the extraction of the potential from the low-frequency asymptotic behavior of the wave field ψ^+. The required formula can be found as follows. First, for sufficiently small k and with mild conditions on V, the wave equation [Eq. (2)] can be solved by iteration. Consequently, the wave field may be expanded as $\psi = \psi_0 + ik\psi_1 + O(k^2)$ and $\psi^{\mathrm{sc}} = \psi_2^{\mathrm{sc}} k^2 + O(k^3)$. We note that the iteration shows that $\psi_0 = 1$ and that

$$\psi_2^{\mathrm{sc}}(\hat{\mathbf{e}},\mathbf{x}) = \int d^3x' G_0^+(k=0,|\mathbf{x}-\mathbf{x}'|)V(\mathbf{x}'). \tag{4}$$

But $G_0^+(k=0,\dots)$ is the Green's function for the Laplacian [set $k=0$ in Eq. (3)]. Consequently, operating on both sides of (4) by Δ yields

$$V(\mathbf{x}) = \Delta\psi_2^{\mathrm{sc}}(\hat{\mathbf{e}},\mathbf{x}). \tag{5}$$

Equation (5) is a crucial result in our approach. The use of low frequencies[8] is a significant departure from methods which focus primarily on the wave-front conditions.[9,10]

If we could compute $\psi_2^{+\mathrm{sc}}(\hat{\mathbf{e}},\mathbf{x})$ from the scattering data, the inverse problem would be solved. Consequently, we look for an equation which relates the wave field and the scattering amplitude. Such an equation is

$$\psi^+(k,\hat{\mathbf{e}},\mathbf{x}) = \psi^-(k,\hat{\mathbf{e}},\mathbf{x}) + ik(2\pi)^{-1}\int d^2\hat{e}' A(k,\hat{\mathbf{e}}',\hat{\mathbf{e}})\psi^-(k,\hat{\mathbf{e}}',\mathbf{x}). \tag{6}$$

Here the scattering amplitude, A, is defined by the large-($x \equiv |\mathbf{x}|$) asymptotic behavior of $\psi^{+\mathrm{sc}}$:

$$\psi^{+\mathrm{sc}}(k,\hat{\mathbf{e}},\mathbf{x}) \equiv A(k,\hat{\mathbf{x}},\hat{\mathbf{e}})e^{-ikx}x^{-1} + O(x^{-2}), \tag{7}$$

where the direction of scattering is $\hat{\mathbf{x}} = \mathbf{x}/x$.

Equation (6) was proved in Ref. 10 for the wave equation. We focus on it because a similar equation was used in the solution of the inverse problem for Schrödinger's equation.[2] Naively (6) is a promising candidate to relate $\psi_2^{+\mathrm{sc}}$ and A. However, Eq. (6) cannot be used directly to determine $\psi_2^{+\mathrm{sc}}$ because A vanishes as k^2 in the limit $k \to 0$. Consequently, Eq. (6) for $k=0$ is merely $\psi_2^+ = \psi_2^-$ which does not directly relate ψ_2^+ to A.

Nonetheless, Eq. (6) is still essential. The problem is that we have not, up until now, exploited the causal structure of the problem, an element which seems essential in exact inverse scattering. This causal structure is clearest in the time domain. We take the Fourier transform (FT) of Eqs. (2) and (6) using

$$u^{\pm}(t,\hat{\mathbf{e}},\mathbf{x}) = (2\pi)^{-1}\int dk\, e^{-ikt}\psi^{\pm}(k,\hat{\mathbf{e}},\mathbf{x}). \tag{8}$$

Similarly, $u^{\pm\mathrm{sc}} \equiv \mathrm{FT}(\psi^{\pm\mathrm{sc}})$. As discussed by Rose and co-workers,[10,11] u^+ is the wave field in the time-domain representation and is generated by an incident δ-function plane wave $\delta(t-\hat{\mathbf{e}}\cdot\mathbf{x})$.

The causal structure of the problem and its relation to (6) will now be sketched. In the absence of a scatterer, $u^+(t,\hat{\mathbf{e}},\mathbf{x})$ would be $\delta(t-\hat{\mathbf{e}}\cdot\mathbf{x})$. Consequently, in this case $u^+(t,\hat{\mathbf{e}},\mathbf{x}) = 0$ for $t < \hat{\mathbf{e}}\cdot\mathbf{x}$. Now consider the case when a scatterer exists, but with $c(\mathbf{x}) < 1$. Since the velocity is less than that of the embedding medium, the initial wave front can only be slowed down. Thus, the first excitation at a point $\mathbf{x}$ due to the incident pulse can only occur at times later than or equal to $t = \hat{\mathbf{e}}\cdot\mathbf{x}$, and for $t < \hat{\mathbf{e}}\cdot\mathbf{x}$ both $u^+(t,\hat{\mathbf{e}},\mathbf{x}) = 0$ and $u^{+\mathrm{sc}}(t,\hat{\mathbf{e}},\mathbf{x}) = 0$. We define $u^- = \mathrm{FT}(\psi^-)$. From the Fourier transform of (2) (see Refs. 10 and 11), it follows that $u^-(t,\hat{\mathbf{e}},\mathbf{x}) = u^+(-t,-\hat{\mathbf{e}},\mathbf{x})$. Consequently, $u^-(t,\hat{\mathbf{e}},\mathbf{x})$ and $u^{-\mathrm{sc}}(t,\hat{\mathbf{e}},\mathbf{x})$ are zero for $t > \hat{\mathbf{e}}\cdot\mathbf{x}$.

Thus, the Fourier transform of (6) can be written for $t > \hat{\mathbf{e}}\cdot\mathbf{x}$ as

$$u^{+\mathrm{sc}}(t,\hat{\mathbf{e}},\mathbf{x}) = (4\pi^2)^{-1}\int_{-\infty}^{\infty} dk\, ik \int_{S^2} d^2\hat{e}' A(k,\hat{\mathbf{e}}',\hat{\mathbf{e}})\psi^-(k,\hat{\mathbf{e}}',\mathbf{x})e^{-ikt}. \tag{9}$$

Here we have used $u^-(t,\hat{\mathbf{e}},\mathbf{x}) = 0$ for $t > \hat{\mathbf{e}}\cdot\mathbf{x}$. Since $u^{+\mathrm{sc}}(t,\hat{\mathbf{e}},\mathbf{x}) = 0$ for $t < \hat{\mathbf{e}}\cdot\mathbf{x}$, Eq. (9) can be integrated in time form $\hat{\mathbf{e}}\cdot\mathbf{x}$ to infinity to obtain ψ_2^{sc} which determines V in Eq. (5).

Equation (2), (5), and (9) are the promised set of self-consistent equations. Their simultaneous solution (V,ψ) solves the inverse problem in the sense that the scattering amplitude generated by $V(\mathbf{x})$ for a given incident direction is identical to the prescribed data.

Little is yet known concerning methods of solving Eqs. (2), (5), and (9). A typical (although possibly naive) method for attempting to find a solution is to iterate the questions. Iterative methods, when they work at all, often converge rapidly. Iterative solutions of self-consistent equations are found in many areas of physics. Examples are the Hartree and Hartree-Fock equations[12] of atomic and molecular physics.

An iterative scheme might proceed as follows. Guess a potential called $V_0(\mathbf{x})$. Then use Eq. (2) to generate a new field, called ψ^{+0} from $V_0(\mathbf{x})$. This field ψ^{+0} is substituted on the right-hand side of (9) and yields an estimate for u^{+0}. Upon Fourier transformation we obtain new estimates ψ^{+0} and ψ_2^{+0}. These are then substituted in Eq. (5) to yield a new estimate for the potential. Call it V_1. Schematically,

$$V_0(\mathbf{x}) \xrightarrow{\text{Eq. (2)}} \psi^{+0} \xrightarrow{\text{Eq. (9)}} \psi^{+0} \xrightarrow{\text{Eq. (5)}} V_1(\mathbf{x}) \rightarrow \dots . \qquad (10)$$

This process can then be iterated. If the iterations converge, a solution to the problem is obtained.

Equation (9) may be inconvenient for numerical calculations since it contains the δ function and possibly other distributions. However, Eqs. (5) and (9) can be combined in a way which avoids this difficulty as we now describe schematically. We define $p^+ \equiv \psi^{+\mathrm{sc}}/k^2$, and its Fourier transform $q^+ \equiv \mathrm{FT}(p^+)$. Then we proceed by (1) subtracting $\exp(ik\hat{\mathbf{e}}\cdot\mathbf{x})$ from both sides of Eq. (6); (2) dividing the resulting equation by k^2; (3) Fourier transformation to the time domain as in Eq. (8); and (4) using the causal condition $q^+(t,\hat{\mathbf{e}},\mathbf{x})=0$, $t<\hat{\mathbf{e}}\cdot\mathbf{x}$. This leads to an equation for q^+ similar to Eq. (9). Using the fact that $\psi_2^{+\mathrm{sc}} = \int_{-\infty}^{\infty} dt\, q^{+\mathrm{sc}}(t,\hat{\mathbf{e}},\mathbf{x})$ together with Eq. (5) one obtains

$$V(\mathbf{x}) = (2\pi^2)^{-1}\mathrm{Re}\Delta\int_0^\infty dk\, e^{-ik\hat{\mathbf{e}}\cdot\mathbf{x}}\int_{S^2} d^2\hat{e}'\, A(k,\hat{\mathbf{e}}',\hat{\mathbf{e}})k^{-2}\psi^{+*}(k,-\hat{\mathbf{e}}',\mathbf{x}). \qquad (11)$$

The self-consistent process now reduces to solving Eqs. (2) and (11) simultaneously. Little is known about obtaining solutions to these equations in the general case. However, an approximate solution of (2) and (11) in the weak-scattering case is consistent with an inversion method based on the Born approximation.[13]

Equation (11) is reminiscent of a number of other formulas that have appeared in the inverse scattering literature. For example, it bears some resemblance to an approximate formula based on the distorted-wave Born approximation.[14] A formula giving the potential in terms of the data and the wave field was also found by Deift and Trubowitz[7] for the case of the one-dimensional Schrödinger equation. They called their equation the trace formula; it is the foundation of their inverse scattering method. For the case of the three-dimensional Schrödinger equation, a formula with the same structure as (11) was derived by Newton.[15]

Some preliminary work has been done to determine if an iterative solution of Eqs. (2) and (11) is feasible. Namely, the numerical calculation of Eq. (11) has been studied for spherically symmetric potentials supposing that the correct ψ^+ is known. Spherical symmetry allows one to perform the integration over angles analytically by expanding in spherical harmonics. We report results for the following test case. Namely, the velocity is set equal to 0.50 for $r<a$ and to 1.00 for $>a$. The wave field and its derivative $d\psi/dr$ are assumed to be continuous at $r=a$. The k integration is obtained by discretization and evaluation of Eq. (11) after expansion in spherical harmonics. Our grid points for the integral are chosen at $ka=0.00$, 0.10, 0.20, ..., 10.00. The velocity profile obtained from the resulting $V(\mathbf{x})$ is shown by the dashed line in Fig. 1. Generally, the agreement between the exact and the computed velocity profiles is quite good. The rounding near the discontinuity and the small oscillations are presumably due to the cutoff in the evaluation of the k integration at $ka=10.00$.

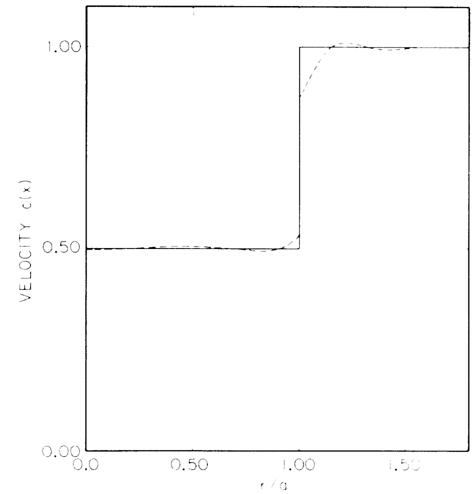

FIG. 1. Test velocity profile (solid line) and reconstructed velocity profile (dashed line).

Several comments are in order. First, the proposed method requires data which depend on only three variables: i.e., the scattering amplitude for one direction of incidence, all directions of scatter, and all wave vectors k. This is in contrast to Newton's exact method[2] for Schrödinger's equation where the required data depend on five variables. Second, in solving the self-consistent equations, the most labor will be required to solve the wave equation. The other equations are quadratures. This is a desirable feature since considerable effort has gone into optimizing solution methods for the wave equation. Finally, a similar set of self-consistent equations can be written down for the three-dimensional inverse scattering problem for Schrödinger's equation. These equations would be the Lippmann-Schwinger equation, Eq. (9) of this Letter, and the equation which relates the potential to the wave front[2,8,10] [see Eq. (3.2) of Ref. 11].

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