PHYSICAL REVIEW LETTERS

VOLUME 59 | 31 AUGUST 1987 | NUMBER 9

Measurement of Longitudinal Coherence Lengths in Particle Beams

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(Received 20 May 1987)

We point out that, on very general grounds, experiments with directionally collimated beams (of neutrons or other particles) incident on passive targets cannot distinguish between their coherent wave-packet structure and the incoherent mixing due, for example, to multiple emitters or to interactions with other particles in the source.

PACS numbers: 03.65.Bz, 03.80.+r

Discussions of atomic and subatomic interactions generally assume that the interacting particles start in wave-packet states which have respectable enough properties so that the usual S-matrix formalism applies. We do not believe that this assumption has led to incorrect results. Nevertheless, it is of some interest to study the wave-packet properties of actual particle beams. Attempts to do this have been reported in the recent literature,[1] and have led to some discussion.[2,3] Of course, wave packets as pure states, or coherent superpositions, are the exception rather than the rule in practice. Most sources have such a statistical character that only a mixture of states can adequately represent our understanding of the beam.

We point out in this Letter that, on very general grounds, experiments with directionally collimated beams (of neutrons or other particles) incident on passive targets cannot distinguish between their coherent wave-packet structure and the incoherent mixing due, for example, to multiple emitters or to interactions with other particles in the source. Our conclusion holds very generally subject to two restrictions for experiments performed with incident unidirectional waves. First, the transverse components of momentum necessary to confine the initial beam must be negligible on the scale of the experiment. That is, an incident momentum-space wave packet $\phi(\mathbf{k})=\phi(\mathbf{e}_3k_z+\mathbf{k}_\perp)$ is replaced by

$$a(\mathbf{k})=\int d\mathbf{k}_\perp\,\phi(\mathbf{e}_3k_z+\mathbf{k}_\perp),$$

where the vector $\mathbf{k}$ in $a(\mathbf{k})$ is one dimensional, $\mathbf{k}=\mathbf{e}_3k_z$, and where we also assume that k_z may be limited to positive values.[4] Second, the target must be describable as a pure state of almost definite energy, or as an incoherent superposition of such pure states.

We illustrate the distinction we have in mind by a simple interferometer experiment performed on a light beam emitted by a gas of excited atoms. For purposes of illustration, we may take for a component of the electric field emitted by each atom

$$\phi_i(x,t)=\begin{cases}a_i\exp\{i[k_i(x-x_i)-ck_i(t-t_i)]-\frac{1}{2}\Gamma[t-t_i-(x-x_i)/c]\} & \text{for } c(t-t_i)\geq x-x_i,\\ 0 \quad \text{for } c(t-t_i)<x-x_i,\end{cases} \tag{1}$$

where x_i and t_i are the place and time of excitation, $1/\Gamma$ is the lifetime of the emitting atom, k_i and ck_i are the wave number and frequency of the emitted line, and a_i is a constant proportional to the dipole strength of the emitting atom.

The interferometer will split the beam coherently into two parts, and bring the parts together with different path lengths corresponding to combining the wave function at two points x_1 and x_2 with $\Delta x = x_2 - x_1$.

The integrated flux, P_i, will be given by

$$P_i = \frac{|a_i|^2}{\Gamma}\left[1+\cos(k_i\Delta x)\exp\left(\frac{-\Gamma|\Delta x|}{2c}\right)\right], \tag{2}$$

and the interference pattern will become blurred beyond a path difference $\Delta x_0 \simeq 2c/\Gamma$, the coherence length of the radiated wave packet.

There is, however, a second effect which also blurs the pattern: The thermal longitudinal motion of the atoms shifts the line according to $k_i = k_0(1+v_i/c)$, leading to an average P

$$P \simeq \frac{|a_0|^2}{\Gamma}\left[1+\cos(k_0\Delta x)\exp\left(-\Gamma\frac{|\Delta x|}{2c}\right)\exp\left(-(\Delta x)^2\frac{\kappa T}{2mc^2}k_0^2\right)\right], \tag{3}$$

where κ is Boltzmann's constant, T the absolute temperature of the gas, and m the mass of the radiating atoms.

Clearly the two effects are, in general, indistinguishable without a theoretical analysis such as we have given. The first factor multiplying the interference term $\cos(k_0\Delta x)$ indeed comes from the wave-packet coherence length; the second reflects the effect of an incoherent superposition of sources. The combined effect is uniquely determined by the wave-number distribution of the beam.

This result is well known as applied to optical phenomena, and is equivalent to the Wiener-Khintchine theorem. (See, for example, Born and Wolf.[5]) We turn next to a general discussion of interference experiments in quantum theory, where we will see that the same result holds.

We may characterize the incident state by a "wave function"

$$\Psi_i = \int d\mathbf{k}\, a(\mathbf{k})\,|\mathbf{k},n\rangle, \tag{4}$$

where n is the initial target state, of energy ϵ, and the $\mathbf{k}$ integration is restricted to a single direction—the direction of propagation of the wave packet.

If Ψ represents a pure state, $a(\mathbf{k})$ is a function. If not (and that is why we have put "wave function" in quotes), the average $\overline{a(\mathbf{k})a^*(\mathbf{k}')}$ must be set equal to the density matrix $\rho(\mathbf{k},\mathbf{k}')$. Note that $\overline{|a(\mathbf{k})|^2} = \rho(\mathbf{k},\mathbf{k})$ is the wave-number spectrum of the incident beam. Of course, the density ρ must also characterize the spin configuration of the beam. We suppress spin variables in the following.

The beam is manipulated by a sequence of mirrors, transmitters, scatterers, etc. (the target). The system emerges in a final state

$$\Psi_f = U\int d\mathbf{k}\, a(\mathbf{k})\,|\mathbf{k},n\rangle = \int d\mathbf{k}\int d\mathbf{k}_f \sum_{n_f} |\mathbf{k}_f,n_f\rangle\langle\mathbf{k}_f,n_f|U|\mathbf{k},n\rangle a(\mathbf{k}), \tag{5}$$

where the U operator, assumed *known* in this analysis, transforms the incident plane-wave state $|\mathbf{k},n\rangle$ into the corresponding outgoing wave eigenstate of the Hamiltonian.

The number of incident neutrons per unit area at the point $\mathbf{x}$ is

$$\frac{dN}{dA} = N\int_{-\infty}^{\infty}(\Psi_i, \mathbf{j}(\mathbf{x},t)\Psi_i)\cdot\mathbf{e}\,dt, \tag{6}$$

where $\mathbf{e}$ is the beam direction, N the total number of incident particles, and $\mathbf{j}(\mathbf{x},t)$ the probability current. We use the formula

$$\langle\mathbf{k}'|\mathbf{j}(\mathbf{x},t)|\mathbf{k}\rangle = \frac{\hbar(\mathbf{k}+\mathbf{k}')}{2m}\exp(i\{(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x} - [\omega(\mathbf{k})-\omega(\mathbf{k}')]t\}), \tag{7}$$

which yields

$$\frac{dN}{dA} = N\mathbf{e}\cdot\int d\mathbf{k}'\, a^*(\mathbf{k}')\int d\mathbf{k}\, a(\mathbf{k})\frac{\hbar(\mathbf{k}+\mathbf{k}')}{2m}\exp\{i(\mathbf{k}-\mathbf{k}')\cdot x\}2\pi\delta(\omega-\omega') \tag{8}$$

$$= 2\pi N\int \overline{|a(\mathbf{k})|^2}\,d\mathbf{k}, \tag{9}$$

according to our restrictions on the nature of the wave packet. The ensemble average

$$\overline{|a(\mathbf{k})|^2} = \rho(\mathbf{k},\mathbf{k}). \tag{10}$$

The number of final particles crossing a detector of area $\delta\mathbf{A}$ is

$$\delta N = N\,\delta\mathbf{A}\cdot\int_{-\infty}^{\infty} dt\,(\Psi_f, \mathbf{j}(\mathbf{x},t)\Psi_f), \tag{11}$$

or

$$\delta N = N\,\delta\mathbf{A}\cdot\int d\mathbf{k}\int d\mathbf{k}'\int d\mathbf{k}_f\int d\mathbf{k}_f'\sum_{n_f}\exp\{i(\mathbf{k}_f-\mathbf{k}_f')\cdot\mathbf{x}\}\frac{\hbar(\mathbf{k}_f+\mathbf{k}_f')}{2m}$$
$$\times\langle\mathbf{k}_f,n_f\,|\,U\,|\,\mathbf{k},n\rangle\langle\mathbf{k}_f'n_f\,|\,U\,|\,\mathbf{k}',n\rangle^*2\pi\delta(\omega(\mathbf{k})-\omega(\mathbf{k}'))\overline{a^*(\mathbf{k}')a(\mathbf{k})} \tag{12}$$

or

$$\delta N = N\,\delta\mathbf{A}\cdot\sum_{n_f}\int d\mathbf{k}_f\int d\mathbf{k}_f'\int d\mathbf{k}\left[\frac{\mathbf{k}_f+\mathbf{k}_f'}{2k}\right]\exp\{i(\mathbf{k}_f-\mathbf{k}_f')\cdot\mathbf{x}\}\langle\mathbf{k}_f,n_f\,|\,U\,|\,\mathbf{k},n\rangle\langle\mathbf{k}_f',n_f\,|\,U\,|\,\mathbf{k},n\rangle^*\rho(\mathbf{k},\mathbf{k}). \tag{13}$$

The energies $\omega(\mathbf{k})$ and $\omega(\mathbf{k}')$ rather than $\omega(\mathbf{k}_f)$ and $\omega(\mathbf{k}_f')$ appear in the matrix element of $(\Psi_f,\mathbf{j}(\mathbf{x},t)\Psi_f)$, since $U\,|\,\mathbf{k},n\rangle$ is an eigenstate of the Hamiltonian with the *incident* energy, and the time dependence of the operator $\mathbf{j}$ is

$$\mathbf{j}(\mathbf{x},t)=e^{iHt/\hbar}\mathbf{j}(\mathbf{x},0)e^{-iHt/\hbar}.$$

Note also that, contrary to its appearance, Eq. (13) does not depend on the location of the origin of the coordinate system since the matrix elements of U cancel out any such dependence.

Thus, just as in the earlier example, the ratio

$$\frac{dN/dA\,|_{\text{final}}}{dN/dA\,|_{\text{incident}}}$$

depends only on the wave-number spectrum, $\overline{|a(\mathbf{k})|^2}=\rho(\mathbf{k},\mathbf{k})$. The wave-number spectrum itself will imply some restrictions on the nature of the wave packets making up the incident beam. For example, a model of the beam as a completely incoherent mixture of individual wave packets, ψ_i:

$$\Psi=\textstyle\sum_i a_i\psi_i, \tag{14}$$

with

$$a_ia_j^*=\delta_{ij}p_i, \tag{15}$$

and

$$\textstyle\sum_i p_i=1 \tag{16}$$

has the property

$$\overline{(q-\bar{q})^2}=\textstyle\sum_i p_i(\Delta q_i)^2+\sum_i p_i(q_i-\bar{q})^2, \tag{17}$$

where q is any observable, q_i its mean value, and $(\Delta q_i)^2$ its variance in the state ψ_i. We note that Eqs. (15) and (16) will always hold for those ψ_i's which diagonalize the density matrix. In addition, they may hold for physical reasons with other sets of wave functions, as in our example above [Eq. (1)]. Equation (17) shows that the ensemble variance of any observable is the sum of the individual variances $(\Delta q_i)^2$ averaged over the ensemble plus the ensemble variance of the individual averages q_i. In particular, with $q=k$, the wave number, we see that the overall variance $\overline{(k-\bar{k})^2}\geq$ average of the individual wave-packet variances $(\Delta k_i)^2$, and hence will in general limit from below the lengths of the individual wave trains in the states ψ_i. Clearly, no upper limit on the lengths of individual wave trains can be deduced from $\rho(\mathbf{k},\mathbf{k})$.

The considerations presented are based on conventional stationary-state scattering analysis, appropriate to analyze any *passive* configuration of an interferometer. Of course, time-dependent apparatus, including time discriminated counting and initial beam chopping, could be used to prepare artificial wave packets and to measure their interference effects. These experiments, however interesting themselves, would not necessarily produce evidence bearing on the state of the neutrons in the incident beam of radiation.

We wish to thank Professor C. G. Shull for arousing our interest in this problem and for a helpful discussion. This work was supported in part by funds provided by the U.S. Department of Energy under Contract No. DE-AC02-76ER03069, and by the National Science Foundation under Grant No. DMR-85-13369.

[1] H. Kaiser, S. A. Werner, and E. A. George, Phys. Rev. Lett. **50**, 560 (1983).

[2] A. G. Klein, G. I. Opat, and W. A. Hamilton, Phys. Rev. Lett. **50**, 563 (1983).

[3] G. Comsa, Phys. Rev. Lett. **51**, 1105 (1983); and its reply by H. Kaiser, S. A. Werner, and E. A. George, Phys. Rev. Lett. **51**, 1106 (1983).

[4] The transverse momentum $k_\perp$ necessary to confine the initial beam is restricted to $k_\perp\cdot L_\perp\gg 1$, where $L_\perp$ is the overall effective transverse dimension of the apparatus. For the $k_\perp$ to be negligible, we must have $k_\perp/k_z\ll 1$ (where k_z is a typical longitudinal wave number) and the transverse spreading $\hbar k_\perp t/m\ll L_\perp$, with t the time of propagation through the apparatus, $t\sim L/V=Lm/\hbar k_z$, where L is the effective longitudinal dimension of the apparatus. The restrictions on $k_\perp$ are then summarized by the inequalities

$$k_zL_\perp/L\gg k_\perp\gg 1/L_\perp,$$

which are clearly consistent provided $k_zL_\perp^2/L\gg 1$, a condition which is very well satisfied for almost all experimental arrangements.

[5] M. Born and E. Wolf, *Principles of Optics* (Pergamon, New York, 1970), 5th ed., p. 504.