# Topological Defects of Wave Patterns

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(Received 26 May 1987)

We identify the defects of waves by means of topological arguments and study them in the framework of Landau-type analysis. It is shown that they correspond to sinks, sources, or dislocations of traveling waves, and to dislocations of standing waves.

PACS numbers: 47.10.+g, 05.45.+b, 47.20.Ky

Phase transitions are generally associated with the breaking of fundamental symmetries, and are usually accompanied by the appearance of defects.[1] Recently, defects in macroscopic systems, such as dislocations and grain boundaries in convective structures, have attracted considerable attention[2-6]; in particular, they seem to play an important role in the transition to turbulence.[7,8] Particularly interesting are defects of wave patterns since they do not have a strict equilibrium analog. Such patterns can be regarded as leading to a *propagative order* in the same way as the appearance of rhythms leads to temporal order and the appearance of periodic patterns leads to spatial order. A lot of interest has been recently devoted to spatiotemporal patterns and associated defects, such as homogeneous oscillatory patterns and their spiral-wave defects,[9] traveling waves in open flows[10] or in confined systems,[11-17] and standing-wave patterns.[18] In this Letter we study topological defects of wave patterns in the framework of phenomenological amplitude equations generalizing Newell's envelope equations[19] via Landau-type symmetry arguments.

We are interested in systems assumed to be invariant under rotations, space and time translations, and parity transformations. We place ourselves in a parameter region where they undergo an instability with finite wave number $k_0$ and frequency $\omega_0$. A typical physical quantity, such as the temperature or the alcohol concentration in a binary mixture, is then expressed as

$$T=\mathrm{Re}\{A(x,y,t)\exp[i(k_0x+\omega_0t)]+B(x,y,t)\exp[i(-k_0x+\omega_0t)]\}+\cdots. \tag{1}$$

The complex order parameters $A$ and $B$ stand for slowly varying envelopes of left- and right-traveling waves propagating in the $x$ direction, and satisfy coupled Landau-Newell–type equations which read in appropriate scaled form:

$$\frac{\partial}{\partial t}A+c\frac{\partial}{\partial x}A=\mu A+(1+i\alpha)\left[\frac{\partial}{\partial x}-\frac{i}{2k_0}\frac{\partial^2}{\partial y^2}\right]^2A+i\epsilon\left[\frac{\partial^2}{\partial x^2}+\zeta\frac{\partial^2}{\partial y^2}\right]A-(1-i\beta)|A|^2A-(\gamma+i\delta)|B|^2A, \tag{2a}$$

$$\frac{\partial}{\partial t}B-c\frac{\partial}{\partial x}B=\mu B-(1+i\alpha)\left[\frac{\partial}{\partial x}+\frac{i}{2k_0}\frac{\partial^2}{\partial y^2}\right]^2B+i\epsilon\left[\frac{\partial^2}{\partial x^2}+\zeta\frac{\partial^2}{\partial y^2}\right]B-(1+i\beta)|B|^2B-(\gamma+i\delta)|A|^2B, \tag{2b}$$

where $\mu$ measures the deviation from the critical situation; $c$, $\alpha$, $\epsilon$, and $\zeta$ describe dispersive effects and are simply related to the imaginary part of $\sigma_{\mathbf{k}}$; $\beta$ and $\delta$ are associated with nonlinear renormalization of the temporal frequency, and $\gamma$ is the competition parameter between traveling and standing waves, corresponding to nontrivial homogeneous solutions of Eqs. (2). The symmetries of the original system determine the form of these equations which are then invariant under the following transformations: $A\to A\exp(-i\Phi)$ and $B\to B\exp(i\Phi)$, which reflects the invariance under space translations; $A\to A\exp(i\Psi)$ and $B\to B\exp(i\Psi)$, which reflects the invariance under time translations; $x\to -x$, $A\to B$, $B\to A$, and $y\to -y$, which reflects the parity symmetry.

Equations (2) possess two types of nontrivial solutions: the traveling waves,

$$A=Q_a\exp[i(\Omega_at+\phi_a)],\quad B=0, \tag{3a}$$

$$A=0,\quad B=Q_b\exp[i(\Omega_bt+\phi_b)], \tag{3b}$$

where $Q_a^2=Q_b^2=\mu$ and $\Omega_a=\Omega_b=-\beta\mu$; and the standing waves,

$$A=Q\exp[i(\Omega t+\phi_a)],\quad B=\exp[i(\Omega t+\phi_b)], \tag{4}$$

where $Q^2=\mu/(1+\gamma)$, $\Omega=-(\beta+\delta)/(1+\gamma)$. The former are stable with respect to spatially homogeneous perturbations when $\gamma>1$, the latter when $-1<\gamma<1$. In both cases, $\phi_a$ and $\phi_b$ are arbitrary phases.

A nonhomogeneous solution corresponding to a left-traveling wave with a wave number slightly different from $k_0$ reads $A=Q_a\exp[i(\Omega_a t-px+\phi_a)]$, $B=0$ where $Q_a^2=\mu-p^2$, and $\Omega_a=pc-\beta\mu-(\alpha-\beta+\epsilon)p^2$. Such a solution is stable with respect to small perturbations when

$$D_\parallel=1+\alpha\beta-2p^2\frac{1+\beta^2}{\mu-p^2}+\epsilon\beta,$$

$$D_\perp=-\frac{p}{k_0}(1+\alpha\beta)+\beta\epsilon\zeta,$$

and

$$a_{yyyy}=\frac{1}{k_0^2}\left[1+\alpha\beta+(1+\beta^2)\frac{(\alpha p-\epsilon\zeta k_0)^2}{2(\mu-p^2)}\right]$$

are strictly positive parameters. In what follows, we consider a range of parameters for which traveling waves are found to be stable.

Equations (2) also possess a more general class of solutions:

$$A=Q_a\exp[i(\Omega_a t-px+\phi_a)],$$
$$B=Q_b\exp[i(\Omega_b t+qx+\phi_b)], \qquad (5)$$

where

$$Q_a^2=\frac{\mu(1-\gamma)-p^2+\gamma q^2}{1-\gamma^2},\quad \Omega_a=-\frac{\beta+\delta}{1+\gamma}\mu+pc-(\epsilon+\alpha)p^2+\frac{p^2(\beta-\gamma\delta)+q^2(\delta-\gamma\beta)}{1-\gamma^2}$$

and $Q_b$ and $\Omega_b$ are obtained by the substitutions $p\to q$ and $q\to p$. They correspond to standing waves when $p=q$. In the following, we place ourselves in a parameter regime where standing waves are stable with respect to small perturbations. This corresponds to $1+\alpha(\gamma\delta-\beta)/(\gamma^2-1)>0$ when $p=q=0$.

Besides homogeneous and quasihomogeneous waves described so far, more singular solutions of Eqs. (2) play a very important role in real-life wave formation. These solutions are the analog of defects in symmetry-breaking phase transitions and are related to the nontrivial topology of the manifold of the stable homogeneous solutions of Eqs. (2).[20] In the following, we describe these defects by means of both topological arguments and numerical experiments performed on Eqs. (2). The numerical simulations have been performed on Cray-1 supercomputer with a spectral code with 80×80 collocation points and a "slaved frog"[21] temporal scheme.

(i) *Traveling-wave defects.*— When $\gamma>1$, the manifold of stable homogeneous states ($\mathcal{M}$) is composed of two disconnected circles parametrized by the phases $\phi_a$ and $\phi_b$ of the left- and right-traveling waves [see Eqs. (3)]. From the nonconnectedness of $\mathcal{M}$, it follows the existence of topologically stable kinklike defects, corresponding to a point defect in one spatial dimension and to a line defect in two dimensions. In what follows, we consider only the cases of a defect line either parallel or perpendicular to the wave vector.[22] In the first case, the kink-type defect connecting ($A=Q_a\exp[i(\Omega_a t-px)]$ $B=0$) as $x\to-\infty$ to ($A=0$, $B=Q_b\exp[i(\Omega_b t+px)]$) as $x\to+\infty$ represents a sink of traveling waves, while the antikink-type one, connecting ($A=0$, $B=Q_b\exp[i\times(\Omega_b t+px)]$) to ($A=Q_a\exp[i(\Omega_a t-px)]$, $B=0$) is associated with a source of traveling waves. In both cases, at the defect's core, $A$ and $B$ are finite (see Fig. 1).

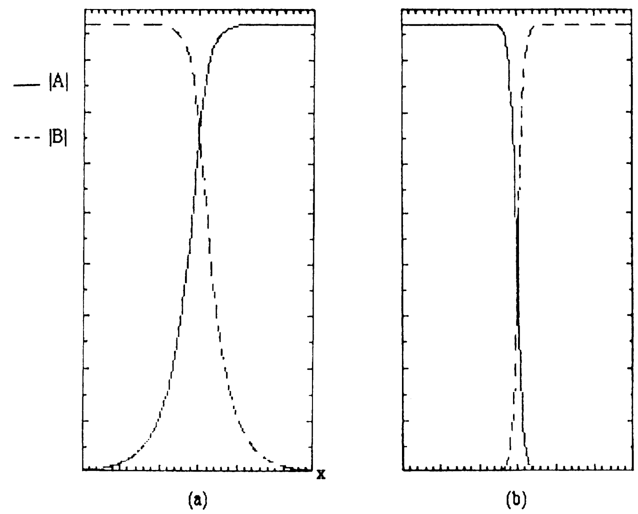


FIG. 1. $|A|$ and $|B|$ as functions of $x$ for (a) a sink and (b) a source of traveling waves.

The asymmetry between the source and the sink observed in the numerical simulation can be easily explained by the symmetry-breaking propagative terms $\pm c(\partial/\partial x)$ and $\pm(1+i\alpha)(i/k_0)(\partial/\partial x)(\partial^2/\partial y^2)$ in Eqs. (2). Far from the core, the defect reaches its asymptotic solutions in an exponentially damped oscillatory wave.[23] We also numerically observe that the final pattern has a unique wave number, independent of initial states. This wave-number selection is clearly due to the presence of defects. Figure 2 shows the temporal behavior of $T$ as defined in (1) in the case of a sink and a source of traveling waves. These topological defects have been recently experimentally identified.[24,25]

The domain wall connecting two traveling waves with slightly different temporal frequencies $\Omega_a$ and $\Omega_b$ moves with a velocity $v_\perp=(\Lambda-\alpha-\epsilon)\delta\Omega/[c+2q(\beta-\alpha-\epsilon)]$ and satisfies the following zig-zag–type equation:

$$(\partial/\partial t)X=v_\perp+D_\parallel(\partial^2/\partial y^2)X-\sigma_1^2(\partial^4/\partial y^4)X+\sigma_2^2[(\partial/\partial y)X]^2(\partial^2/\partial y^2)X, \qquad (6)$$

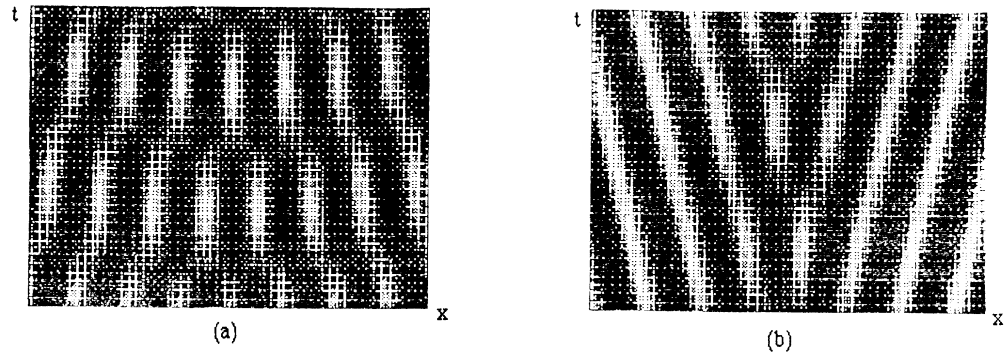


FIG. 2. Diagrams $(x,t)$ for (a) a sink and (b) a source of traveling waves. Dark areas correspond to maxima of $T$ [see Eq. (1)].

where the perpendicular diffusion coefficient is given by

$$D_{\|} = [(p+q)/2k_0](1+\alpha\Lambda)+\epsilon\zeta\Lambda,$$

$$(\delta\Omega) = \Omega_a - \Omega_b,$$

and $\Lambda$, $\sigma_1^2$, and $\sigma_2^2$ are given constants, which can be computed by standard methods.[26,27]

"Zipper"-like[12] defects which correspond to domain walls perpendicular to the wave vector are also found to be numerically stable solutions of Eqs. (2).

From the nontrivial topology of each of the two circles of $\mathcal{M}$, it also follows the existence of dislocations. These topological defects are characterized by their winding number $N$. Physically, they correspond to the insertion of $N$ extra critical wavelengths. As usual, at the defect's core, the order parameter $A$ (and, respectively, $B$) vanishes (see Fig. 3). We find numerically that the dislocation of the left-traveling wave is stable with respect to the right-traveling–wave perturbations, which in fact justifies the use of an equation involving only $A$, corresponding to (2) with $B=0$ (see Ref. 16). Because of the advective term $c(\partial A/\partial x)$, the dislocation has a relative motion with respect to the host wave. This dislocation motion can be pinned by the underlying propagative structure, but this effect cannot be studied in the framework of an amplitude analysis.[28]

(ii) *Standing-wave defects.*— When $-1<\gamma<1$, the coexistence of right- and left-traveling waves leads to dynamically stable standing waves. The order parameters $A$ and $B$ are both finite and consequently the manifold of stable homogeneous states is a torus parametrized by the phases $\phi_a$ and $\phi_b$ [see Eq. (4)]. Standing-wave defects are thus characterized by two topological charges $(q_1,q_2)$, corresponding to the number of extra wavelengths added on left- and right-traveling waves. The elementary defects (1,0) and (0,1), are called, respectively, "left" and "right" dislocations. At the core of a left dislocation, the envelope $A$ of the left-traveling wave vanishes and thus, since the envelope $B$ of the right-traveling wave remains finite, one observes right propagation. As for traveling waves, the presence of defects induces a strict wave-number selection: Far from the core, one observes numerically a quasistanding wave, which belongs to the family of solutions given by Eq. (5),

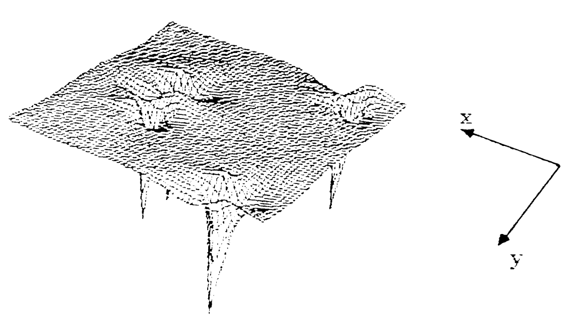


FIG. 3. $|A|$ as a function of $x$ and $y$ showing four traveling-wave dislocations corresponding to $N=1$.

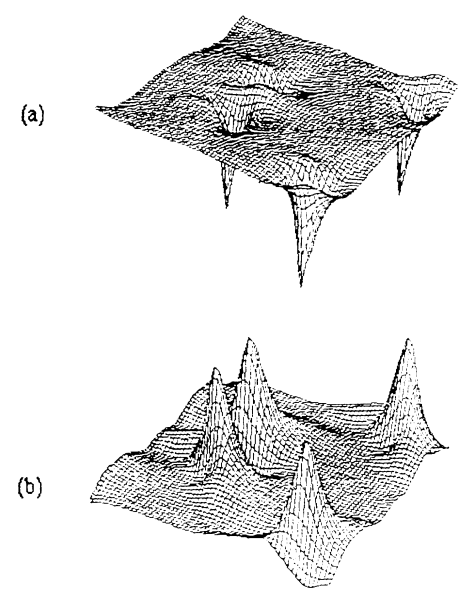


FIG. 4. (a) $|A|$ and (b) $|B|$ as functions of $x$ and $y$ showing four standing-wave dislocations, corresponding to $q_1=1$ and $q_2=0$.

with $p \neq 0$ and $q = 0$. We also note that when $|A|$ vanishes, $|B|$ reaches its maximum value (see Fig. 4). As noticed for traveling waves, the core of the left dislocation moves through the box in the $x$ direction because of the advective term $c(\partial A/\partial x)$.

The motivation of this Letter has been twofold. First, we have described wave defects in the framework of Landau-type analysis. They correspond to sinks, sources and dislocations of traveling waves, and right and left dislocations of standing waves. They have a topological origin just as defects in equilibrium physics. These new defects are particularly interesting since they are described by nonvariational equation. Second, our aim was to stimulate experimental studies of such objects, especially in the case of standing waves.

We thank F. Guerin for her careful reading of this manuscript. This work has been supported by an Action Thématique Programmé of the Centre National de la Recherche Scientifique, the Direction des Recherches et Etudes Techniques, and the Centre de Calcul Vectoriel pour la Recherche, where the numerical simulations have been performed. We also acknowledge the National Center for Atmospheric Research, which is supported by the U.S. National Science Foundation.

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