Nonlinear O(N) σ Model: A Theory with Hidden U(1) Gauge Invariance

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The (1+1)-dimensional O(N) nonlinear σ model is represented as a gauge theory with a U(1) gauge group. It is remarked that in the quantized version of the theory a dynamical Higgs phenomenon takes place. On the basis of this fact, we speculate on the existence of a similar gauge-symmetry–breaking mechanism for vector fields for which no additional fields are required.

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Nonlinear σ models for various reasons have attracted much attention during the last two decades. They were first introduced to model the low-energy limit of a theory with spontaneous symmetry breakdown.[1] In the course of time it was realized that (1+1)-dimensional σ models have striking qualitative similarities with QCD.[2] The (partial) list of common features of both field theories includes renormalizability, asymptotic freedom, scale invariance on the classical level broken by an anomaly in quantum theory, and low-energy behavior which is dominated by local condensates. Because of this and due to the existence of a rigorous solution,[3] σ models serve as a laboratory for various nonperturbative techniques, for example, the $1/N$ expansion, operator-product expansion, and low-energy theorems. Recently, the interest in these models was revived in the context of string theory[4] where they appear in the classical limit.

We shall concentrate on the simplest (two dimensional) O(N)-invariant σ model which is a free field theory for a multiplet of fields $\sigma=(\sigma_1,\ldots,\sigma_n)$ satisfying the constraint $\sigma^2=1$. Any constrained system can be represented in two equivalent ways[2,5]: (1) via introduction of the Lagrange multiplier into the Lagrangean, and (2) by a gauge-invariant Lagrangean in which gauge transformations are generated by constraints. The nonlinear σ model is usually represented in the first form while electrodynamics (which is another constrained system) is usually represented in the second form.

In this Letter, we find for the σ model an explicit form of Lagrangean invariant under U(1) local gauge transformations. We remark that in this theory the phenomenon of dynamical mass generation (in the context of gauge theory, the dynamical Higgs mechanism) occurs.

Our line of reasoning will be parallel to that leading to the description of electrodynamics in terms of potentials, and therefore we shall closely follow the analogy between these systems in this Letter. Classical electrodynamics is described by the Hamiltonian

$$H(E,A)=\tfrac{1}{2}E_i^2+\tfrac{1}{4}(\nabla_i A_j-\nabla_j A_i)^2, \tag{1}$$

which commutes (has zero Poisson bracket) with the Gauss constraint

$$C(E)=\nabla_i E_i. \tag{2}$$

The potentials A_i are conjugate to the canonical variables E_i. The constraint function $C(E)$ generates the U(1) group of (time-independent) canonical transformations

$$\delta A_i=\{A_i,\chi(x)C(E)\}=\nabla_i\chi(x), \tag{3}$$

which leaves the Hamiltonian $H(E,A)$ invariant.

For the σ model the Hamiltonian which leads to correct equations of motion may be taken to be

$$H(\sigma,\pi)=\tfrac{1}{2}\sigma^2\pi_i\left(\delta_{ij}-\frac{\sigma_i\sigma_j}{\sigma^2}\right)\pi_j+\tfrac{1}{2}\nabla\sigma_i\nabla\sigma_i, \tag{4}$$

which commutes with the constraint

$$C(\sigma)=\tfrac{1}{2}(\sigma^2-1). \tag{5}$$

The corresponding transformations are

$$\begin{aligned}\delta\sigma_i&=\{\sigma_i,\chi(x)C(\sigma)\}=0,\\ \delta\pi_i&=\{\pi_i,\chi(x)C(\sigma)\}=\chi(x)\sigma_i.\end{aligned} \tag{6}$$

The constraint can be incorporated by introduction of the additional term $h=\alpha C$ to the Hamiltonian, where α is a Lagrange multiplier. In electrodynamics

$$h=A_0 C(E), \tag{7}$$

where A_0 plays the role of α. In the σ model the corresponding term is

$$h=\alpha C(\sigma). \tag{8}$$

Now the Hamiltonian $H+h$ is invariant under the full (time-dependent) gauge group U(1), provided that the Lagrange-multiplier field transforms as

$$\delta\alpha=-\dot{\chi}. \tag{9}$$

In electrodynamics, the gauge-invariant Lagrangean $L(A,\dot{A})=\tfrac{1}{4}F^2$ is obtained by Legendre transformation of $H+h$ expressing E via A and $\dot{A}$ from the Hamilton equation. Let us do the same for the σ model. The Hamilton equations

$$\dot{\pi}_i = \sigma_i \pi^2 - \pi_i(\sigma\pi) + (\alpha + \nabla^2)\sigma_i \tag{10}$$

permit one to express σ in terms of π_i, $\dot{\pi}_i$, and α:

$$\sigma_i = \frac{1}{\alpha + \nabla^2 + \pi^2}\left[\dot{\pi}_i + \pi_i \frac{\pi\dot{\pi}}{\alpha + \nabla^2}\right]. \tag{11}$$

The Lagrangean is

$$L = \tfrac{1}{2}\{[\dot{\pi}\pi^2/(\alpha+\nabla^2)(\alpha+\nabla^2+\pi^2)] + [\dot{\pi}^2/(\alpha+\nabla^2+\pi^2)] + \alpha\}. \tag{12}$$

This Lagrangean is invariant under the time-dependent gauge transformation

$$\pi_i \rightarrow \pi_i + \frac{\chi}{\alpha+\nabla^2+\pi^2}\left[\dot{\pi}_i + \pi_i \frac{\pi\dot{\pi}}{\alpha+\nabla^2}\right], \qquad \alpha \rightarrow \alpha - \dot{\chi}. \tag{13}$$

In terms of the "potentials" π_i and α the gauge invariant Lagrangean L, in contrast to electrodynamics, is nonlocal, and Lorentz invariance is not manifest. This can be traced from the locality and Lorentz covariance properties of the constraints in both cases. In electromagnetism the constraint equation is the fourth component of a vector equation and is differential, while in the σ model the constraint equation is a scalar equation and does not contain derivatives.

In accordance with the general procedure one must fix the gauge. Since the transformation (13) is nonlinear it is more convenient to work in the Hamiltonian formalism. The momentum conjugate to the constraint $C(\sigma)$ is

$$G = \pi\sigma/\sigma^2. \tag{14}$$

In order to fix the (ghostless) gauge, one has to add to the Hamiltonian Eq. (4) the term $h_{\text{g.f.}}$ whose Poisson bracket with C is a nontrivial function of C and G only. In order to make the resulting Hamiltonian as simple as possible we choose the following gauge-fixing term:

$$h_{\text{g.f.}} = \frac{1}{2}(\sigma^2 - 1)\left(\pi^2 - \frac{(\pi\sigma)^2}{\sigma^2}\right) - \frac{1}{2}\frac{(\pi\sigma)^2}{\sigma^2}. \tag{15}$$

Indeed

$$\{h_{\text{g.f.}}, C\} = \pi\sigma = G(C-1). \tag{16}$$

The resulting Hamiltonian is

$$H + h + h_{\text{g.f.}} = \tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\sigma)^2 + \tfrac{1}{2}\alpha(\sigma^2 - 1). \tag{17}$$

It coincides with the standard Hamiltonian[2] used to quantize the model. Thus the standard quantization is, in fact, ghostless. In this gauge the Lagrangean for "potentials" simplifies to

$$L = \frac{1}{2}\frac{\dot{\pi}^2}{\alpha - \nabla^2} - \frac{1}{2}\pi^2 + \frac{1}{2}\alpha, \tag{18}$$

with gauge condition

$$\frac{\pi\dot{\pi}}{\alpha + \nabla^2} = 0. \tag{19}$$

The quantum σ model in $(1+1)$ dimensions is a nontrivial solvable[3] quantum field theory. Perturbatively the spectrum of the theory has no energy gap, i.e., it has massless excitations. However, the $1/N$ expansion (confirmed by the exact solution) shows that the true (nonperturbative) spectrum has no zero modes, but rather N massive particles with mass[2]

$$m^2 = \mu^2 \exp\left(-\frac{4\pi}{f(\mu)}\right), \tag{20}$$

where $f(\mu)$ is a running coupling constant.

This energy gap in the $1/N$ expansion is associated with the expectation value of the field α. This seemingly contradicts Elitzur's theorem, since according to Eq. (13), α is a gauge-variant quantity. However, the $1/N$ expansion is performed with the Hamiltonian of Eq. (17) which includes the gauge fixing term of Eq. (15). After the gauge fixing α is no longer gauge variant. In this gauge α is equal to the following quantity:

$$\alpha = \tfrac{1}{2}\sigma^2\pi_i\left(\delta_{ij} - \sigma_i\sigma_j/\sigma^2\right)\pi_j - \tfrac{1}{2}\nabla\sigma_i\nabla\sigma_i, \tag{21}$$

which is invariant under the gauge transformation Eq. (6). This expression can be easily derived from the equations of motion that follow from the Hamiltonian Eq. (17) using the constraint and the gauge condition $C = \sigma^2 - 1$, $G = \pi\sigma/\sigma^2 = 0$.

Viewing the σ model as a gauge theory, it is interesting to note that this mass generation is a complete analog of mass generation by the dynamical Higgs mechanism (or as it is sometimes called dynamical gauge-symmetry breakdown).[6,7] Indeed, in the Higgs model the field Φ, which (before the gauge is fixed) is a gauge-variant quantity, acquires a nonzero vacuum expectation value. The mass gap is generated in the spectrum of the theory by means of this vacuum expectation value. Analogously in the σ model the field α (which before gauge fixing also has nontrivial transformation properties under the gauge group) acquires a vacuum expectation value, thus generating an energy gap in the spectrum.

The difference between the two cases is that, while in the Higgs model mass generation can be seen already on the classical level, in the σ model the effect is more delicate and can be discovered only after quantum corrections are taken into account. In this respect the model is similar to the Schwinger model.[8]

The existing mechanism of gauge-symmetry breaking in the theory of electroweak interactions is considered by many physicists to be unsatisfactory. The main reason is the hierarchy problem inherent in theories with fundamental scalars. Even worse, recent lattice calculations indicate that SU(2) as well as U(1) gauge theories coupled to Higgs fields are trivial.[9]

In spite of great efforts no realistic alternative mechanism for giving mass to gauge fields was found. In all these attempts the cure has been sought by the introduction of additional fields not required by phenomenology (e.g., technifermions, Higgs superpartners). It is amusing to contemplate another possibility based on our experience with the nonlinear σ model.

Indeed in the $O(N)$ σ model there are $N-1$ fundamental fields, all massless in perturbation theory. However the exact spectrum of excitations contains N massive, in place of $N-1$ massless, particles. The main purpose of the Higgs mechanism is to convert a massless vector field which has two polarizations into a massive vector which has three polarizations.[10] The σ model indicates that it is not possible to achieve this without the introduction of any additional fields.

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[7]The term gauge-symmetry breakdown is somewhat inadequate. There can be, of course, no real breakdown of gauge symmetry in a non-Abelian theory since by construction the only physical states in Hilbert space are those invariant under gauge transformations. For example, in the non-Abelian Higgs model, the statement that the field Φ^i acquires vacuum expectation value has no meaning unless one fixes the gauge in which one works. After the gauge is completely fixed, no gauge invariance is left and Φ^i becomes gauge invariant.

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[10]The suggestion that the number of excitations exceeds the number of elementary fields does not contradict the unitarity of a theory. The exactly solvable σ model provides a striking example of this phenomenon. In technicolorlike theories the unitarity is restored with the help of a "would-be" composite Goldstone boson which eventually decouples from the theory [see, for example, J. Cornwall and R. Norton, Phys. Rev. D **8**, 3338 (1973)]. A similar mechanism may be present also in pure Yang-Mills theory.