Gravitational Stability of Local Strings

R. Gregory
Department of Applied Mathematics and Theoretical Physics, Cambridge CB3 9EW, United Kingdom
(Received 5 January 1987; revised manuscript received 3 April 1987)

The full coupled gravity-string field equations are considered, and they are used to show that a general local string will have an asymptotically conical structure. For the case of the Abelian Higgs model with U(1) gauge invariance, the gravitational field of a simple local string to first order in $G\eta^2$ is exhibited. Then a C-energy argument is used to suggest stability at this linearized level.

PACS numbers: 98.80.Cq, 04.20.Jb, 11.17.+y

There has been a lot of interest recently in the cosmological implications of cosmic strings. These are one of a family of so-called topological defects which can arise during the evolution of the early Universe. (See, e.g., Vilenkin[1] for a review.)

In describing particle interactions, one is often interested in the behavior of some Higgs field ϕ that is possibly coupled to some gauge fields A_μ. The evolution of these fields may be described by a Lagrangean L which has an associated symmetry group G. We are concerned with the case where the field potential energy at low temperatures has the form of a $\lambda_0\phi^4$ interaction with spontaneous symmetry breaking. Then, during the development of the Universe, there could be a phase transition to a spontaneously broken symmetry state, where the vacuum acquires a nonzero expectation value η, thus leaving a residual symmetry group H. If the vacuum manifold (which is isomorphic to G/H) is nonsimply connected, then it is highly probable that cosmic strings will form.[2]

Of particular interest are the gravitational fields of these strings, for, although they are locally weakly gravitating, it is not immediately obvious that the effect of an infinite string (containing an infinite amount of energy) will be equally small. For example, in the case of an infinite tube of low-density pressureless dust, the circumstances of circles of arbitrarily large radii become arbitrarily small. Clearly we need to consider the full coupled gravitational-Higgs-gauge equations of motion to analyze the asymptotic (i.e., far from the string) structure of our space-time. While the questions of existence and uniqueness of solutions have been addressed recently by Garfinkle,[3] following earlier work reviewed in Ref. 1, neither a detailed discussion of boundary conditions nor a stability analysis has been carried out.

Let us consider the idealized situation of an infinite straight static string. Because this system exhibits cylindrical symmetry, we may write the line element in the form[4]

$$ds^2 = e^{2(\gamma-\psi)}(dt^2 - dr^2) - e^{2\psi}dz^2 - \alpha^2 e^{-2\psi}d\theta^2,$$

where γ, ψ, and α are functions of r only. Since I intend to produce nonsingular solutions, I shall impose the boundary conditions of elementary flatness on the symmetry axis, i.e.,

$$\gamma(0) = \psi(0) = 0,$$

and $\alpha \sim r$, as $r \to 0$. Thus space-time along the center of the string is locally flat, and we avoid making any statement on the asymptotic structure of the space-time.

Because our system is assumed cylindrically symmetric, the string fields are z and t independent; thus the only contribution to the z-z and t-t components of the energy-momentum tensor is via the term proportional to the metric, and hence $T^0{}_0 = T^z{}_z$. As mentioned in the introduction, we take λ_0 to be the self-interaction coupling constant, and η the symmetry-breaking scale. It then proves useful to introduce the dimensionless variables

$$\rho = \sqrt{\lambda_0}\eta r, \quad \tilde{\alpha} = \sqrt{\lambda_0}\eta\alpha, \quad \epsilon = 8\pi G\eta^2,$$
$$E = T^0{}_0/\lambda_0\eta^4, \quad -P_\rho = T^r{}_r/\lambda_0\eta^4, \quad -P_\theta = T^\theta{}_\theta/\lambda_0\eta^4.$$

These have been chosen so that the new energies and pressures will have typical order of magnitude 1 (at most), and the radius of the string likewise. ϵ is the parameter which represents the local gravitational strength of the string and will be assumed small. (On physical grounds, ϵ is typically of order 10^{-6}.)

We find that the gravitational-field equations (as stated in Ref. 4) simplify to the following:

$$\tilde{\alpha}'' = -\epsilon\tilde{\alpha}e^{\gamma}(E - P_\rho), \tag{1}$$

$$(\tilde{\alpha}\gamma')' = +\epsilon\tilde{\alpha}e^{\gamma}(P_\rho + P_\theta), \tag{2}$$

$$\tilde{\alpha}'\gamma' = \epsilon\tilde{\alpha}e^{\gamma}P_\rho + \tfrac{1}{4}\tilde{\alpha}\gamma'^2, \tag{3}$$

$$\gamma = 2\psi, \tag{4}$$

where $\gamma' = d\gamma/d\rho$, etc. We also have the equation of motion for the fields:

$$P_\rho' + (P_\rho - P_\theta)\tilde{\alpha}'/\tilde{\alpha} + \tfrac{1}{2}\gamma'(P_\rho + P_\theta) + \gamma' E = 0. \tag{5}$$

Locally, at least, this system of equations plus boundary conditions determines a unique solution. However, we require a somewhat stronger statement upon the asymptotic structure of the space-time. I will show that, provided that the energy density satisfies certain falloff conditions (stated below) and that the string is suitably weakly gravitating, the space-time is asymptotically conical. First, I note that if we make no assumptions other than dominant energy and nonsingularity of the space-time ($\tilde{a} > 0$) for $\rho > 0$,

$$0 < \tilde{a} < \rho, \quad \rho > 0. \tag{6}$$

Second, if we regard (3) as a quadratic equation for γ', regularity at the axis singles out the negative root, giving

$$\gamma' = 2\tilde{a}'[1-(1-\epsilon e^{\gamma} P_\rho \tilde{a}^2/\tilde{a}'^2)^{1/2}]/\tilde{a}, \tag{7}$$

which may be rewritten as

$$\gamma' = 2[\tilde{a}' - f^{1/2}(\rho)]/\tilde{a},$$

where $f(\rho) = \tilde{a}'^2 - \epsilon e^{\gamma} P_\rho \tilde{a}^2$. Note that our boundary conditions at the axis imply $f(0)=1$. In order to preserve the sign choice in (7), we need to prove that $f(\rho) > 0$ for all ρ.

Assume the contrary, i.e., that there exists an $R > 0$ at which f has its first zero. Now, by continuity of γ, there exists a $\Delta < R$ such that $e^{\gamma} < 2$ on $[0,\Delta]$. We first estimate $\tilde{a}'$ on $[0,\Delta]$ using (1) and (6):

$$\tilde{a}' = 1 - \epsilon \int_0^\rho \tilde{a} e^{\gamma}(E - P_\rho) d\rho \geq 1 - \epsilon A \rho^2, \tag{8}$$

where $A = \sup(E - P_\rho)$ on $[0,\Delta]$. Then $f \geq (1-\epsilon A\rho^2)^2 - 2\epsilon B\rho^2$ where $B = \sup|P_\rho|$ on $[0,\Delta]$. Since $f > 0$ on $[0,\Delta]$, the elementary inequality $(1-x)^{1/2} \geq 1 - |x|$ for $x < 1$ implies

$$\gamma' \leq 2\epsilon e^{\gamma} |P_\rho| \tilde{a}/\tilde{a}' \leq 4\epsilon\rho B/(1-\epsilon A\rho^2).$$

Therefore $\gamma' = O(\epsilon)$ on $[0,\Delta]$ for $\Delta \simeq 1$, but we see that we may choose $\Delta \simeq 1$ without violating $f > 0$. Equation (7) applied on $[\Delta, R]$ gives $\gamma' \leq 2\tilde{a}'/\tilde{a}$ and, hence, $e^{\gamma} \leq 2(\rho/\Delta)^2$. Similarly, on $[\Delta, R]$,

$$\tilde{a}' \geq 1 - \epsilon A \Delta^2 - \int_\Delta^R 2\epsilon\rho(\rho/\Delta)^2(E - P_\rho) d\rho.$$

Provided that E, P_ρ, and P_θ fall off sufficiently fast $[O(1/\rho^5)]$ as $\rho \to \infty$, we have that $\tilde{a}' \geq 1 - O(\epsilon)$, and hence $f = 1 - O(\epsilon)$, at $\rho = R$, thus contradicting the original assumption about R. We may now use $\gamma' \leq 2\tilde{a}'/\tilde{a}$ on $[\Delta, \infty]$ to conclude that

$$\int_o^\infty e^{\gamma}(E - P_\rho) d\rho < \infty;$$

hence,[5] the asymptotic solution to (1) is

$$\tilde{a} \sim A\rho + B, \quad \rho \to \infty,$$

and that in fact $\gamma \to C$ as $\rho \to \infty$.

For a local string we expect E, P_ρ, and P_θ to decay exponentially, so that our falloff conditions are easily satisfied. We can conclude the our space-time does indeed look asymptotically conical with a deficit angle proportional to $1 - Ae^{-C}$. Note that this behavior cannot persist to the axis: The effect of the matter is to smooth out the apex of the cone, so that space-time becomes flat at the axis. The spatial $r-\theta$ sections could therefore be likened to snub-nosed cones.

So far we have been considering only string solutions in general. To be specific, let us consider a local string formed in the theory described by the U(1)-gauge-invariant Lagrangean

$$L = D_\mu \phi^\dagger D^\mu \phi - \tfrac{1}{4} f_{\mu\nu} F^{\mu\nu} - \tfrac{1}{4}\lambda_0(\phi^\dagger\phi - \eta^2)^2. \tag{9}$$

Our string is thus a general-relativistic generalization of the Nielsen-Olesen vortex.[6]

We shall consider only the simple string

$$\phi = \eta X(r) e^{i\theta},$$

since we suspect that higher-order strings will be unstable.[7] Setting e as the gauge-coupling constant, we may then write[6]

$$A_\mu = (1/e)[P(r) - 1]\nabla_\mu \theta.$$

Then, in terms of the new fields P and X, the Lagrangean becomes

$$L = -\lambda_0 \eta^4 \left[e^{-\gamma} X'^2 + \frac{e^{\gamma} X^2 P^2}{\tilde{a}^2} + \frac{P'^2}{\tilde{a}^2} + \frac{(X^2-1)^2}{4} \right]. \tag{10}$$

Note that I have set $2e^2 = \lambda_0$, so that the width of the gauge part of the string is the same as that of the scalar part.

From observations of the isotropy of the microwave background radiation, and other limits based on gravitational radiation,[8] we are provided with an upper bound on our string parameters, viz., $G\eta^2 \leq 10^{-6} \ll 1$. Therefore we may expand the fields as an asymptotic series in powers of ϵ, i.e.,

$$\tilde{a} = \sum_0^\infty \epsilon^n \tilde{a}_n,$$

etc., and use an iterative procedure to find $\tilde{a}$, γ, X, and P.

To zeroth order (flat space), $\tilde{a}_0 = \rho$, $\gamma_0 = 0$, and we know[6] that the solutions to the equations of motion (5) have exponentially decaying X, P, and hence E, P_ρ, and P_θ. To first order, we find

$$\tilde{a} = \left[1 - \epsilon\int_0^\rho \rho(E_0 + P_{\rho 0}) d\rho\right]\rho + \epsilon\int_0^\rho \rho^2(E_0 + P_{\rho 0}) d\rho,$$

$$\gamma = -\epsilon \int_0^\rho \rho P_{\rho 0} d\rho.$$

Thus, to first order in ϵ, the deficit angle of our snub-nosed cone is

$$\delta\theta = 2\pi(1 - \tilde{a}' e^{-\gamma}) = 2\pi\epsilon \int_0^\infty \rho E_0 d\rho$$
$$= 8\pi G \int_0^\infty 2\pi r T^0{}_0 dr,$$

i.e., $\delta\theta = 8\pi G\mu$, where μ is the flat-space energy density of the string. Note that the fortuitous cancellation of the radial stresses means that this is in agreement with previous results[9] which assumed that these stresses vanished.

Suppose, however, that we try to perform a similar analysis for the corresponding U(1) global string, i.e., a vortex solution with Lagrangean

$$L = \nabla_\mu \phi^\dagger \nabla^\mu \phi - \tfrac{1}{4}\lambda_0(\phi^\dagger\phi - \eta^2)^2$$
$$= -\lambda_0 \eta^4 [e^{-\gamma} X'^2 + e^{\gamma} X^2/\tilde{\alpha}^2 + (X^2-1)^2/4]$$

in the previous notation. Then we find

$$P_\rho = +X'^2 e^{-\gamma} - X^2 e^{\gamma}/\tilde{\alpha}^2 - (X^2-1)^2/4,$$
$$P_\rho + P_\theta = -(X^2-1)^2/2 \le 0. \qquad (11)$$
$$E - P_\rho = X^2 e^{\gamma}/\tilde{\alpha}^2 + (X^2-1)^2/4.$$

Hence (2) and (6) imply that γ will diverge at least as fast as $-C\ln\rho$ as $\rho \to \infty$, where C is a positive constant. Notice that we have already lost our asymptotically conical structure. The assumption of nonnegativity of $\tilde{\alpha}$, which implies $\tilde{\alpha}'' \le 0$, also implies that $\tilde{\alpha}'$ is nonnegative; in particular, we see that $\tilde{\alpha}$ is bounded away from zero for large ρ. Equation (8) then gives us some necessary conditions for nonnegativity of $\tilde{\alpha}'$, viz.,

$$\int_0^\infty (X^2 e^{2\gamma}/\tilde{\alpha})\,d\rho < \infty,$$
$$\int_0^\infty [\tilde{\alpha} e^{\gamma}(X^2-1)^2/4]\,d\rho < \infty, \qquad (12)$$

and, hence,

$$\int_0^\infty [X^2 e^{2\gamma}/\tilde{\alpha}^2]\,d\rho < \infty, \quad \int_0^\infty [e^{\gamma}(X^2-1)^2/4]\,d\rho < \infty.$$

Therefore by the previous argument, $\alpha \sim A\rho + B$ as $\rho \to \infty$.

However, for a vortex solution $X \to 1$ as $\rho \to \infty$ at least as fast as some negative power of ρ; this, together with (12), suffices to ensure that $\tilde{\alpha} e^{\gamma} P_\rho = O(\rho^{-\nu})$ for some $\nu > 1$ as $\rho \to \infty$. Thus to leading order in ρ, (3) becomes

$$\tilde{\alpha}'\gamma' = \tfrac{1}{4}\tilde{\alpha}\gamma'^2,$$

which cannot be satisfied since $\gamma' < 0$. This shows that the assumption that $\tilde{\alpha} > 0$ cannot be true. Thus these global strings do not have a well-behaved asymptotic structure; this is not entirely surprising, as even their flat-space energy density per unit length is divergent.

Within this "linearized" approximation, we can examine the stability of the local string. I shall argue stability on energetic grounds, using a definition of energy appropriate to cylindrically symmetric systems—the C energy.[4] I choose to apply a relativistic energy analysis (rather than Newtonian), for the simple reason that in the analogous problem of stability of an electromagnetic vortex, C energy correctly predicts stability, whereas a Newtonian analysis indicates instability.[4,10]

The total C energy per unit length of our system is

$$E_c = (1/4G)[\gamma(\infty) - \ln\alpha'(\infty)].$$

To first order, we obtain

$$E_c = (\epsilon/4G)\left[\int_0^\infty \rho E_0\,d\rho\right].$$

Because the string fields die off exponentially fast, this integral has an effective cutoff radius at the width of the string. At this linearized level, we can regard the C energy as being the sum of a "matter" part and a "gravitational" part. In the static case, the matter part is simply the energy density per unit length, the gravitational part exterior to the string does not contribute since the space-time exterior to the string is locally flat, and the contribution of the geometry inside the string appears as a second-order effect. We expect this split to hold in the time-dependent case. Thus *at* this linearized level, we expect perturbations which contribute to the C energy to be perturbations to the geometry exterior to the string core, and perturbations to the string fields within the core.

For our static solution, the contribution of the geometry exterior to the string core is zero, and applying Thorne's C-energy minimum principle to annuli surrounding the string core (of arbitrarily large radii), we see that the C-energy contribution of any finite but arbitrarily large perturbation to the exterior geometry will be nonnegative.

Now consider the contribution due to the string fields. This has the form

$$E_S = \frac{\epsilon}{4G}\left[\int_0^1 \rho E_0\,d\rho\right]$$
$$= \frac{\epsilon}{4G}\int_0^1\left[\rho X_0'^2 + \tfrac{1}{4}\rho(X^2-1)^2 + \frac{X_0^2 P_0^2}{\rho} + \frac{P_0'^2}{\rho}\right]d\rho$$
$$\ge \frac{\epsilon}{4G}\int_0^1 (P_0'^2/\rho)\,d\rho.$$

Because our field configuration satisfies the classical variational equations of motion, it sits at a local minimum energy configuration; we therefore deduce that our static-string configuration is sitting at a local C-energy minimum and is therefore stable to small perturbations. However, it may be possible that a large radial perturbation could collapse the string to zero width forming a conical singularity. By causality, this process could only occur in a finite time if a finite amount of C energy were involved.

To mimic such a collapse scenario, we shall replace ρ by $f(\rho,t)$ where f is monotonic in ρ for fixed t, and to preserve the boundary conditions, we require $f(0,t)=0$ and $f'(0,t)=1$. To mimic radial collapse in time T, we set $f(\rho,0)=\rho$, and $f(\rho,t)\to 0$ as $t\to T$, but f is otherwise arbitrary. Under such a transformation, we note that

$$\int_0^1 (P_0'^2/\rho)\,d\rho \to \int_0^{f(1,t)} (1/f)(dP_0/df)^2\,df = \int_0^1 (\rho/ff')(P_0'^2/\rho)\,d\rho \to \infty \quad \text{as } t \to T.$$

Now the contribution of the matter fields to the C energy consists of a kinetic part and a potential part, each of which is nonnegative; hence, the total C energy per unit length of our perturbed system will be greater than the contribution of the potential of the matter fields. However, this is bounded below by the integral of $P_0'^2/\rho$ which becomes unboundedly large as the collapse proceeds. We deduce that an infinite total C energy per unit length would be involved in cylindrical radial collapse, and therefore such collapse is prohibited within linearized theory.

By considering the full set of field equations for a static cylindrically symmetric string system, I have shown that the surrounding space-time is asymptotically conical, provided that the energies and pressures decay rapidly enough.

For the particular case of the Abelian Higgs model with U(1) gauge invariance, I have exhibited a local string solution to first order in $G\eta^2$, and find that the asymptotic deficit angle is $8\pi G\mu$. Application of the same techniques to the corresponding U(1) global string shows that there is no regular asymptotic structure.

Finally, a C-energy argument shows that, within a linearized theory, such local strings are resistant to radial collapse.

I would like to thank J. M. Stewart and R. H. Brandenberger for their interest and useful suggestions in this work, which was carried out during the tenure of a Science and Engineering Research Council studentship.

[1]A. Vilenkin, Phys. Rep. **121**, 265 (1985).

[2]A. Albrecht and N. Turok, Phys. Rev. Lett. **54**, 1868 (1985); T. W. B. Kibble, J. Phys. A **9**, 1387 (1976); T. Vachaspati and A. Vilenkin, Phys. Rev. D **30**, 2036 (1984).

[3]D. Garfinkle, Phys. Rev. D **32**, 1323 (1985).

[4]K. S. Thorne, Phys. Rev. **138**, B251 (1965).

[5]R. Bellman, *Stability Theory of Differential Equations* (McGraw-Hill, New York, 1953).

[6]H. B. Nielsen and P. Olesen, Nucl. Phys. **B61**, 45 (1973).

[7]E. B. Bogomolnyi, Yad. Fiz. **24**, 861 (1976) [Sov. J. Nucl. Phys. **24**, 449 (1976)].

[8]D. Bennett, Phys. Rev. D **33**, 872 (1986); R. Brandenberger and N. Turok, Phys. Rev. D **33**, 2182 (1986); C. Hogan and M. Rees, Nature (London) **311**, 109 (1984); N. Kaiser and A. Stebbins, Nature (London) **310**, 391 (1984); Ya. B. Zeldovich, Mon. Not. Roy. Astron. Soc. **192**, 663 (1980).

[9]W. Hiscock, Phys. Rev. D **31**, 3288 (1985); B. Linet, Gen. Relat. Gravitation **17**, 1109 (1985); J. Richard Gott, III, Astrophys. J. **288**, 422 (1985); A. Vilenkin, Phys. Rev. D **23**, 852 (1981).

[10]M. A. Melvin, Phys. Lett. **8**, 65 (1964), and Phys. Rev. **139**, 225 (1965).