# Operator Regularization of Green's Functions

D. G. C. McKeon[1,2] and T. N. Sherry[2]
[1]Department of Applied Mathematics, University of Western Ontario, London, Ontario, Canada N6A 5B9
[2]Department of Mathematical Physics, University College Galway, Galway, Ireland
(Received 21 July 1986)

Operator regularization in background-field quantization facilitates the use of a perturbative expansion due to Schwinger to compute Green's functions to all orders. The procedure is distinct from the usual Feynman technique. No explicit divergences are encountered. We illustrate with a $\phi_6^3$ theory and discuss applications to QED, supersymmetry, axial models, and quantum gravity.

PACS numbers: 11.10.Ef, 11.10.Gh

We propose a new calculational procedure for computing Green's functions in background-field quantization. Our procedure is different from the usual Feynman perturbation expansion—we carry out the Gaussian functional integral over the quantum fields *prior to* expanding the kernel in terms of the background fields, as we outline below. A further difference between the two procedures is the approach to regularization. Rather than regulating the initial Lagrangean, we regularize the generating functional prior to the background-field expansion by regularizing the determinants and inverses of operators which occur in the generating functional after the path integral has been evaluated. For this reason we call the procedure operator regularization.

A distinct advantage of our procedure is that we do not introduce any new parameter into the action which would interfere with the invariance of the action under a symmetry (as, for example, dimensional regularization interferes with supersymmetry). An unexpected bonus of this procedure is that we never encounter any infinities. Operator regularization ensures that all Green's functions are finite; we speculate that this is a general feature of operator regularization.

In general we split a field $\phi_i$ into the sum of a classical part $f_i$ and a quantum part $h_i$, namely

$$\phi_i(x)=f_i(x)+h_i(x). \tag{1}$$

Initially we restrict our attention to Lagrangeans of the form

$$\mathcal{L}=\tfrac{1}{2}h_iM_{ij}(f)h_j+\frac{1}{3!}a_{ijk}(f)h_ih_jh_k+\frac{1}{4!}b_{ijkl}h_ih_jh_kh_l. \tag{2}$$

The Euclidean generating functional $Z$ is now evaluated. In performing the functional integral over the quantum fields $h_i$, we follow a procedure distinct from that used to generate Feynman diagrams. We do not separate $M_{ij}(f)$ into a part $M_{ij}^{(0)}$ independent of $f_i$, whose inverse becomes the Feynman propagator, and a part $M_{ij}^{(1)}(f)$ at least linear in $f_i$ that is absorbed into Feynman vertices. Instead, we perform the functional integration on the full bilinear part of the Lagrangean $M_{ij}(f)$ and later rely on the expansion in powers of $f_i$ given in Eq. (8) below to extract the contribution to $Z$ from the Green's function we wish to evaluate. Consequently, we find the Euclidean generating functional

$$Z(f_i,J_i)=s\det{}^{-1/2}[M_{ij}(f)]\exp\left[\int dx\left(\frac{1}{3!}a_{ijk}(f)\frac{\delta^3}{\delta J_i\,\delta J_j\,\delta J_k}+\frac{1}{4!}b_{ijkl}\frac{\delta^4}{\delta J_i\,\delta J_j\,\delta J_k\,\delta J_l}\right)\right]$$
$$\times\exp\left\{-\frac{1}{2}\int dx[J_iM_{ij}^{-1}(f)J_j]\right\}. \tag{3}$$

As the field $\phi_i$ may be either Bose or Fermi, it follows that $M_{ij}(f)$ is, in general, a supermatrix.

The fundamental quantity we regularize is the logarithm of an operator:

$$\ln A\equiv-\lim_{s\to 0}\frac{d^n}{ds^n}\left(\frac{s^{n-1}}{n!}A^{-s}\right)\quad(n=1,2,\dots). \tag{4}$$

For reasons that will become apparent below, no divergences are encountered if the integer $n$ is greater than or equal to the number of "loop momentum integrals" encountered in the computation of the Green's function under examination. Without loss of generality, we can choose $n$ to be equal to the number of "loops." From (4) we obtain

$$s\det A\equiv\exp(s\operatorname{tr}\ln A)=\exp\left\{s\operatorname{tr}\left[-\lim_{s\to 0}\left(\frac{d^n}{ds^n}\frac{s^{n-1}}{n!}\right)A^{-s}\right]\right\}, \tag{5a}$$

and

$$\Gamma(N)A^{-N}\equiv(-1)^{N+1}\frac{d^N}{dA^N}\ln A=\lim_{s\to0}\frac{d^n}{ds^n}\left(\frac{s^{n-1}}{n!}\frac{\Gamma(s+N)}{\Gamma(s)}A^{-s-N}\right). \tag{5b}$$

Before substituting (5a) and (5b) into (3), we rewrite $A^{-s}$ and $A^{-s-N}$ in (5a) and (5b) using

$$A^{-\lambda}=\frac{1}{\Gamma(\lambda)}\int_0^\infty dt\,t^{\lambda-1}e^{-At}. \tag{6}$$

To one-loop order with only Bose external fields, operator regularization reduces to ζ-function regularization.[1] Green's functions can now be computed to any order with use of the following general guidelines: (i) For Green's functions at the $m$-loop level, use (3) to obtain the appropriate skeleton expansion. (ii) At the one-loop level, we are faced with a superdeterminant $s\det M$ which is regulated by means of (5a) and (6). (iii) At the $m$ ($>1$) loop level, we are faced with expressions containing strings of inverses of operators $A^{-1}B^{-1}\cdots$ regulated by means of (5b) and (6):

$$A^{-1}B^{-1}\cdots=\lim_{s\to0}\left(\frac{d^m}{ds^m}\frac{s^m}{m!}A^{-s-1}B^{-s-1}\cdots\right). \tag{7}$$

(iv) The regulated skeleton expansion now contains factors of the form $\exp[-(A_0+A_1)t]$ and $s\,\mathrm{tr}\exp[-(A_0+A_1)t]$, where $A_0$ is independent of $f$, and $A_1$ is at least linear in $f$. These factors are now expanded in powers of $f$:

$$s\,\mathrm{tr}e^{-(A_0+A_1)t}=s\,\mathrm{tr}\left[e^{-A_0t}+(-t)e^{-A_0t}A_1+\frac{(-t)^2}{2}\int_0^1du\,e^{-(1-u)A_0t}A_1e^{-uA_0t}A_1+\dots\right] \tag{8}$$

and similarly for $\exp[-(A_0+A_1)t]$.[2] This expansion in powers of $f$ allows us to identify by inspection the terms contributing to any particular Green's function. (v) Such terms can be simplified[2] by the insertion of complete sets of momentum states $\int d^np\,|p\rangle\langle p|$ into the matrix elements of operators. The resulting integrals are standard.

To illustrate, let us apply operator regularization to $\phi^3_{6-2\epsilon}$ theory. *To one-loop order*,

$$Z^{(1)}(f)=\det{}^{-1/2}(p^2+\lambda f)\quad(p=-i\partial). \tag{9}$$

Regulation of (9) by use of (5a) and (6) gives

$$Z^{(1)}(f)=\exp[\tfrac{1}{2}\zeta'(0)], \tag{10}$$

where

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^\infty dt(t^{s-1}\mathrm{tr}e^{-(p^2+\lambda f)t}). \tag{11}$$

Applying the expansion of (8)-(11) in $n=6-2\epsilon$ dimensions and keeping only those terms bilinear in $f$ yields

$$\zeta_{ff}(x)=\frac{1}{2\Gamma(s)}\int d^np\,f(p)f(-p)\int_0^\infty dt\,t^{s+1}\int\frac{d^nq}{(2\pi)^n}\int_0^1du\,e^{-[q^2+u(1-u)p^2]t}. \tag{12}$$

The integrals in (12) are standard, and we find

$$\zeta_{ff}(s)\cong\frac{\lambda^2}{2}\int\frac{d^np}{(4\pi)^3}f(p)f(-p)\left(-\frac{p^2}{6}\frac{s}{s+\epsilon}[1+\epsilon(\tfrac{8}{3}-\gamma+\ln4\pi-\ln p^2)+s(\tfrac{8}{3}-\ln p^2)\right). \tag{13}$$

From (13) we note that

$$\lim_{s\to0}\frac{d}{ds}\left[\lim_{\epsilon\to0}\zeta_{ff}(s)\right]\neq\lim_{\epsilon\to0}\left[\lim_{s\to0}\frac{d}{ds}\zeta_{ff}(s)\right]. \tag{14}$$

The left-hand side of this inequality, which corresponds to operator regularization of $\phi^3_6$ theory, yields a finite Green's function. The right-hand side of this inequality, which corresponds to the use of dimensional regularization in the Schwinger expansion for $\phi^3_6$ theory, gives precisely the same value as the dimensionally regulated Feynman diagram for this Green's function, including the pole term.

*To two-loop order*, the regulated one-particle-irreducible skeleton expansion obtained from (3) is

$$Z^{(2)}(f)=\lim_{s\to0}\frac{d^2}{ds^2}\frac{s^2}{2!}\frac{\lambda^2}{2!3!}\int d^6x\,d^6y[\langle x|(p^2+\lambda f)^{-s-1}|y\rangle]^3. \tag{15}$$

The parts of $Z^{(2)}$ bilinear in $f$ are

$$Z_{ff}^{(2)} = \lim_{s\to 0}\left\{\frac{d^2}{ds^2}\frac{s^2}{8}\frac{\lambda^4}{(2\pi)^{12}}\int d^6p\, f(p)f(-p)\int d^6k\, d^6q \right.$$
$$\times\left[\frac{\Gamma(s+2)}{\Gamma(s+1)}\int_0^1 du\, u\int_0^1 dv\left(\frac{1}{q^2}\frac{1}{(q+k+p)^2}\right)^{s+1}\frac{1}{[(1-u)k^2+u(k+p)^2]^{s+3}}\right.$$
$$+\frac{\Gamma^2(s+2)}{\Gamma^2(s+1)}\int_0^1 du_1\, du_2\left(\frac{1}{(q+k+p)^2}\right)^{s+1}\frac{1}{[(1-u_1)k^2+u_1(k+p)^2]^{s+2}}$$
$$\left.\left.\times\frac{1}{[(1-u_2)q^2+u_2(q+p)^2]^{s+2}}\right]\right\}. \quad (16)$$

The two "loop integrations" over $q$ and $k$ lead to a double pole at $s=0$. As a consequence of the choice of $n$ in (5b) equal to the number of loops, the result obtained for $Z_{ff}^{(2)}$ from (16), after all the integrals have been evaluated and the limit of $s$ going to zero is taken, contains no uv divergence. The corresponding two-loop two-point Green's function is, consequently, finite.

We have used operator regularization to compute a wide variety of Green's functions. In particular, we have applied our technique to the one-loop vacuum polarization in Yang-Mills theory in the Honerkamp gauge, the one-loop two- and three-point Green's functions in four- and $n$-dimensional QED, the two-loop vacuum polarization in QED, the one-loop graviton correction to the spinor propagator, the one-loop two-point function in the superfield Wess-Zumino model, and all one-loop two- and three-point functions in the component Wess-Zumino model. In all cases our results have respected the relevant Ward identities, and no explicit divergences are ever encountered. Details will be provided elsewhere.

The models for which these computations are carried out are more sophisticated than the simple model used here for illustration. Consequently, a number of interesting features occur in these applications of operator regularization.

In the $\phi^3_{6-2\epsilon}$ model, $M_{ij}(f)$ is an ordinary matrix. When external fermion fields occur, however, $M_{ij}(f)$ is a supermatrix. The simplest application of operator regularization is the most obvious—regularize $s\det M$ and $M^{-1}$ (the supermatrix inverse) directly with use of Eqs. (4)–(6). This is straightforward.

An alternative application of operator regularization in this case would be to use either one of the two "component" representations of $s\det M$ and $M^{-1}$ defined in the work of Van Nieuwenhuizen.[3] Such an application is less straightforward as inverses of operators appear throughout the various expressions (e.g., the representation of $s\det M$), and each of these inverses must be regularized with use of Eqs. (4)–(6). Nevertheless, we have shown that precisely the same results are found as when we treat the supermatrix directly.

Some of the models considered above are gauge-theory models. As in conventional perturbation theory, computations may be simpler in one gauge than in another. In the covariant Honerkamp gauge with $\alpha=1$, the computations are as straightforward as in the $\phi^3_{6-2\epsilon}$ model. For $\alpha$ arbitrary, however, inverses of operators automatically occur in all calculations. These are regularized with use of Eqs. (4)–(6).

We have applied our procedure for computing Green's functions to the computation of the $VVA$ and $AAA$ three-point Green's functions in an Abelian model which contains external vector and axial-vector fields and quantum fermion fields, and is invariant under vector gauge transformations. We have also applied our computation to the on-mass-shell Green's function associated with the decay of the supercurrent into a vector and a spinor in $N=1$ super-Yang-Mills theory. Straightforward computation in these cases leads to expressions which respect gauge invariance in the vector vertices; consequently, the anomaly resides in the axial-vector vertex in the one case and in the supercurrent vertex in the other case, both of which we have computed.

Explicit calculations establish that the advantages of operator regularization persist even when applied to theories normally considered unrenormalizable. For example, one-loop calculations in $\phi^6_4$ and, more importantly, in quantum gravity are uv finite and symmetry preserving. Operator regularization does not, of course, cure other pathological problems afflicting quantum field theories, such as the breakdown of unitarity in a four-fermion theory.

Finally, we note that the dimensionful constant $\mu^2$ that usually occurs in Green's functions can be introduced by a rescaling of $t$ in (6) so that $\tau=\mu^2 t$ is dimensionless.

We would like to thank the Natural Science and Engineering Research Council of Canada and the University of Western Ontario and the University College Galway academic development funds for financial support. Participants of the 1985 Workshop in Quantum Field Theory stimulated much of this investigation.

[1]A. Salam and J. Strathdee, Nucl. Phys. B **90**, 203 (1975); J. Dowker and R. Critchley, Phys. Rev. D **13**, 3224 (1976); S. Hawking, Commun. Math. Phys. **55**, 133 (1977).

[2]J. Schwinger, Phys. Rev. **82**, 664 (1951).

[3]P. Van Nieuwenhuizen, Phys. Rep. **68**, 189 (1981).