Relativistic, Many-Particle Lagrangean for Electromagnetic Interactions

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It is shown that the Fokker-Wheeler-Feynman model of a system of massive, charged particles interacting via forces of electromagnetic origin can be rewritten to yield a physically acceptable relativistic many-particle Lagrangean. Contrary to the postulates of Wheeler and Feynman, the model satisfies causality and can be generalized to include arbitrary forces. It also allows a simple description of radiation reaction and, as well, may provide the basis for any comparable modeling of other systems.

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Herein, an exact, physically acceptable classical relativistic Lagrangean is derived for a system of charged, massive point particles interacting via action-at-a-distance forces of electromagnetic origin without self-interactions. This type of modeling of such systems is considered at the present time for a number of reasons. First, it is a standard approach, with a long and successful history, to both the classical and quantum descriptions of nonrelativistic systems for which a classical description exists. However, the search for a similar approach to relativistic systems is incomplete (e.g., Llosa[1]). Second, such a modeling should yield a precise definition of particle-particle and particle–external-field interactions and hence suffice for these cases. In fact, our initial interest in this problem arose because calculated relativistic effects in applied-field phenomena of many-electron atoms indicated an inadequacy of current theories (e.g., one-particle[2] and model[3] approximations). Third, it is expected that an action-at-a-distance approach should complement a field-theoretic approach (e.g., Llosa,[1] p. iv, and Sazdjian[4]) and hence lead to a fuller understanding of relativistic theories. Also it may be more straightforwardly applied to many-particle systems. Fourth, the electromagnetic case is being used as a prototype for a similar modeling of two-particle systems (e.g., spinless particles[5] and quarkonium[6-8]). However, no fundamental criteria seem to exist by which to determine the relativistic particle-particle interactions, and they are inserted in *ad hoc* ways. To complete such approaches it is necessary to determine the requisite basic properties of such models, and the electromagnetic problem is a natural first choice. Particular features of the electromagnetic problem are displayed below that have a number of implications and that may be useful in clarifying and unifying some of the basic concepts in predictive relativistic mechanics (e.g., Llosa[1] and Bel[9]) and other types of modeling (e.g., Sasdjian[4,10]).

One might think that the electromagnetic interaction is completely understood; however, this is, in fact, not the case. The Breit Hamiltonian[11] is the most practical model that currently exists for atomic systems. However, in the first relativistic correction to the interparticle interaction, the particle velocities are replaced by their Dirac velocities rather than by their momentum dependence. Thus the substitution $\mathbf{p} \rightarrow \mathbf{p} - (e/c)\mathbf{A}$ cannot be made and, for applied external fields, terms of the desired order are missing. Now the Breit Hamiltonian is derived from the Darwin Lagrangean,[12] which in turn is obtained from the intuitively correct physical model of the one-particle Lagrangean consisting of the free-particle relativistic Lagrangean minus the interaction energy of the particle with the retarded Liénard-Wiechert fields of any other particles. The interaction is expanded in its $1/c$ power series and truncated at the $1/c^2$ term. It is known[13,14] that any attempt at including higher-order terms runs into difficulties. Simply put, the $1/c^3$ term cannot be symmetrized with respect to interchange of particles by the addition of a total time derivative; that is, a *divergence*. Following Darwin, we have extended the expansion to order $1/c^{10}$ using the Maple symbolic algebra system available at the University of Waterloo[15] and have verified that the above difficulty with the $1/c^3$ term is also present for each odd-powered term up to this order. One can only conclude that the initial model employed by Darwin is incomplete.

The only other existing model for the present problem is the Fokker-Wheeler-Feynman (FWF) model[16,17] (see Cramer[18] for a recent review and references). Although originally formulated as an alternative theory of electromagnetism and discarded as such, it has all the ingredients necessary for the type of model being sought. However, along with a number of attractive features, it has, as presented, a number of physically unacceptable features and, hence, has also not been retained for the latter. We will show that the FWF model can, in fact, be recast in a form which is devoid of any difficulties and yields a physically acceptable many-particle Lagrangean. Generally, expressions are written for two particles; however, the extension to many particles is straightforward.

The attractiveness of the FWF model resides in its origins. It is derived from a Lorentz-invariant action integral containing both time reversal and particle symmetry, and the equations of motion are obtained by the standard extremal principle. The effective Lagrangean for particle 1 interacting with particle 2 is given as[16,17]

$$L_1 = -m_{10}c^2/\gamma_1 - \tfrac{1}{2}(V_{12}^R + V_{12}^A), \tag{1}$$

where m_{10} is particle 1's rest mass, c is the velocity of light, $\gamma_1 = (1-u_1^2/c^2)^{-1/2}$ with $\mathbf{u}_1 = d\mathbf{r}_1/dt$ being the usual velocity of particle 1, boldface denoting a three-vector, and

$$V_{12}^\alpha = \frac{s^\alpha q_1 q_2 \tilde{p}_1 \cdot \tilde{p}_2^\alpha}{\epsilon_0 c m_{10} \gamma_1 \tilde{p}_2^\alpha \cdot (\tilde{r}_1 - \tilde{r}_2^\alpha)} \tag{2}$$

subject to

$$c(t_1 - t_2^\alpha) = s^\alpha |\mathbf{r}_1 - \mathbf{r}_2^\alpha|. \tag{3}$$

In the above, the symbols have their usual meaning; the tilde designates a four-vector, $\alpha = R$ with $s^R = +1$ signifies that the quantity is evaluated at the retarded time and yields the interaction of particle 1 with the retarded Liénard-Wiechert field of particle 2, and $\alpha = A$ with $s^A = -1$ gives the corresponding advanced quantities. Notice that Eq. (1) is in the correct form[19,20] to make the Euler-Lagrange equations of motion properly Lorentz covariant.

The difficulties arise in interpreting Eq. (1). In spite of the original action integral being symmetric in particle interchange, the interaction in Eq. (1) is not[21] and, hence, cannot be used as a two-particle interaction in this form. The dependence on future times implies a nonconservation of causality which was simply postulated away.[16,17] As a result one is faced with the paradox of "discontinuous" forces, which was again postulated away. This situation is physically unacceptable in the desired model.

Kerner[21] has almost resolved these difficulties. He has shown that the $1/c$ power-series expansion of the interaction in Eq. (1) contains only even powers; hence the difficulties with the odd-powered terms in the Darwin approach are nonexistent in the FWF model, and that, in the power-series form, the interaction can be symmetrized by the addition of a divergence. Finally, he has shown the existence of a generalized Hamiltonian, total linear momentum, and total angular momentum that are all constants of the motion. These results are promising; however, the physical interpretation is lost in the mathematics. Now a divergence in closed form is obtained by defining

$$F = \tfrac{1}{2}\int_t^{t_2^A} V_{21}^R(t')dt', \tag{4}$$

where V_{21}^R is defined by Eqs. (2) and (3) with the appropriate relabeling. Noting that

$$\frac{dt_2^A}{dt} = \frac{m_{20}\gamma_2^A[\tilde{p}_1 \cdot (\tilde{r}_2 - \tilde{r}_2^A)]}{m_{10}\gamma_1[\tilde{p}_2^A \cdot (\tilde{r}_1 - \tilde{r}_2^A)]}, \tag{5}$$

and taking account of the arguments, one finds

$$dF/dt = \tfrac{1}{2}(V_{12}^A - V_{21}^R). \tag{6}$$

Thus adding Eq. (6) to Eq. (1) yields

$$L_1' = -m_{10}c^2/\gamma_1 - \tfrac{1}{2}(V_{12}^R + V_{21}^R). \tag{7}$$

The two Lagrangeans, Eqs. (1) and (7), describe the same system and differ only by a divergence. To decide which one, if either, is physically meaningful, recall that it is well known in classical mechanics that, in general, divergences are arbitrary functions and any attempt at attributing a physical meaning to them can lead to nonsensical conclusions. The situation here is analogous. However, there are no *a prior* arguments for making a choice and one must decide which Lagrangean leads to physical absurdities. Equation (1) does, while Eq. (7) does not. The interaction in Eq. (7) is symmetric in interchange of the two particles and hence can be taken to represent a two-particle interaction. It is true that we must modify, at a relativistic level, our intuitive concept of the interaction of electromagnetic origin between two particles, but, once the Darwin approach is rejected, that is unavoidable. Equation (7) indicates that there must be an internal self-consistency in the interaction, not that the future affects the present. Notice that the Euler-Lagrange equations of motion cannot be used to make a distinction between the two Lagrangeans because the divergence, Eq. (6), satisfies the Euler-Lagrange equations *identically*. Thus the advanced potential, V_{ij}^A, is *equivalent* to the retarded potential, V_{ji}^R. Since Eq. (7) depends only on past times, the system is causal. Thus the FWF theory is in actual fact a predictive relativistic theory. The implications are rather far reaching. Finally, the paradox of "discontinuous" forces does not occur and Eq. (7) can be generalized, a necessary property of any reasonable model, by adding any additional interactions in any physically acceptable way. This completes the recasting of the FWF model in a form that retains all of its attributes but none of the absurdities originally attributed to it.

As an example of the simplicity of the present interpretation it is shown that radiation reaction follows straightforwardly from the equations of motion and their solutions, a necessary property of such a model. Thus, the model is restored purely to one of a system of particles and their interactions. From Eq. (7) the Lagrangean for particle 1 interacting with the rest of the Universe is

$$L_1' = -m_{10}c^2/\gamma_1 - \textstyle\sum_j' V_{1j}^R + \tfrac{1}{2}\sum_j'(V_{1j}^R - V_{j1}^R), \tag{8}$$

where j is summed over the rest of the Universe, and the prime means $j \neq 1$. The first interaction term is the usual interaction of particle 1 with the retarded field of the rest of the Universe, as derived from Maxwell's equations and used to define the driving fields in any experiment. The second interaction term is missing from the Darwin approach; it must arise from the self-consistent coexistence of the Universe from all past time to the present and it must be included for a complete and correct description of the system. The $1/c$ power-series expansion of this second interaction yields

$$L_1' = -m_{10}c^2/\gamma_1 - \sum_j{}' V_{1j}^R + \sum_j{}' \frac{2q_1 q_j}{3\epsilon_0 c^3}\mathbf{u}_1 \cdot \dot{\mathbf{u}}_j - df/dt + O(1/c^5), \quad (9)$$

where f contains terms of order $1/c^2$, $1/c^3$, and $1/c^4$. The self-consistent interaction is seen to be small, of order $1/c^3$; the correction to the above form being of order $1/c^5$. The Euler-Lagrange equation of motion from Eq. (9) to order $1/c^3$ in the self-consistent interaction is

$$\frac{d}{dt}(m_{10}\gamma_1 \mathbf{u}_1) = \mathbf{F}^R - \sum_j{}' \frac{2q_1 q_j}{3\epsilon_0 c^3}\ddot{\mathbf{u}}_j. \quad (10)$$

$\mathbf{F}^R$ is the usual retarded Lorentz force on particle 1 due to the rest of the Universe. Both terms are instantaneous as the information has already come from the past and particle 1 simply responds to it. No information from the future is involved and there are no response time delays. However, in order to obtain the solution for the motion of particle 1, we need, in principle, to know the motions of all of the particles in the Universe, as is inherent in such a model. Not only is this impractical, it would also not correspond to reality because the real Universe is not described solely by electromagnetic forces. The simplest approach is to infer the required information from observation. First, although the Universe appears to be neutral, this aspect must be used to define the system and gives no information about the motions. Second, despite large variations in distribution, on average, any finite volume of the Universe is also neutral. This we call local charge neutrality and quantify it by

$$\sum_j q_j \mathbf{r}_j = \mathbf{0}. \quad (11)$$

Here the sum over j can be over any volume. Another interpretation of Eq. (11) is that, on average, any finite volume of the Universe does not have a net dipole moment. Any deviations will appear as fluctuations. Notice also that Eq. (11) is not a constraint on the system and, hence, cannot be used in the Lagrangean. Rather, it describes the present solution (state) of the Universe. If the Universe were in a different state this term would be different. Including particle 1 in the sum in Eq. (11), differentiating thrice with respect to time, and substituting into Eq. (10) yields the usual radiation reaction force [e.g., Jackson,[19] Eq. (17.8)]. As a matter of interest, the condition that the Universe be a perfect absorber [Rohrlich,[22] Eq. (7.25)] is satisfied to the same order; that is to $1/c^4$. This is readily seen from Kerner's[21,23] work.

At this point, the foundations, interpretation, and properties of a relativistic Lagrangean for a system of massive, charged particles interacting by forces of electromagnetic origin have been firmly established. Thus we have the basis from which to proceed to other calculations, both classical and quantum mechanical, for closed systems and systems interacting with external sources. Although the $1/c$ power-series expansion of the present Lagrangean results in an infinite-order Lagrangean and it is not yet known how to quantize such systems exactly,[23,24] work done on the Breit Hamiltonian indicates that a meaningful approximate quantization can be achieved. Finally, we now have an acceptable prototype for other systems.

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