New Test of the Reduced-Width–Amplitude Distribution

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The assumption that the distribution of reduced partial-width amplitudes is multivariate Gaussian is tested by separate measurement of width and amplitude correlations. The data are transformed to a representation in which the amplitude correlation is zero, and the width correlation is compared directly with the predicted value of zero. The average width correlation for 21 data sets is $\bar{r}_w = -0.01$, providing the most direct and assumption-free verification that the global amplitude distribution is multivariate Gaussian. The significance of this result for current studies of quantum chaos is discussed.

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The role of quantum chaos in physics and searches for quantum systems which display chaotic behavior have been topics of much discussion recently.[1,2] One technique which has been utilized is to study the quantum analog of a system whose classical behavior is known to be chaotic; the canonical example of this approach is the quantization of Sinai's billiard by Berry.[3] Bohigas, Giannoni, and Schmit[4] studied the level fluctuations of the quantum Sinai's billiard and concluded that they were "fully consistent" with the predictions of the Gaussian orthogonal ensemble (GOE) of random-matrix theory.[5] They offered the conjecture that this relationship between time-reversal–invariant systems whose classical analogs are chaotic and the GOE is universal. Additional results for different systems have thus far supported this conjecture.[6] This connection between spectral rigidity in a quantum system and the regular or chaotic nature of the corresponding classical system has been investigated by Berry[7] and further discussed by Wintgen.[8]

The most extensive set of experimental data used for comparison with GOE predictions has been a collection of high-quality neutron and proton resonance data. The fluctuation properties of this ensemble of energy levels have been studied with a variety of measures[9,10]; the data show both the short-range and long-range order as well as higher-order correlations required by GOE.

Since nuclear energy levels provide the best experimental data thus far for tests of random-matrix theory, it is important to test as many predictions of the theory as possible for these data. In this Letter, we wish to examine the strength fluctuations of nuclear levels. Under the assumptions of GOE, reduced widths in a single channel should obey the Porter-Thomas distribution,[11] and reduced-width amplitudes in multiple channels may be correlated and should follow a multivariate Gaussian distribution.[12]

For reduced widths the Porter-Thomas distribution seems to describe a variety of neutron[13] and proton[14] resonance data. Since the Porter-Thomas distribution for reduced widths is equivalent to a Gaussian distribution for reduced-width amplitudes, agreement of the Porter-Thomas distribution with experimental data seemed to verify the Gaussian assumption. However, Harney[15] has shown that very large samples (larger than available) are necessary for a precise test of the Porter-Thomas distribution. Thus an alternative method of testing the Gaussian assumption is desired. Because one cannot measure the absolute signs of the amplitudes, a test which depends on relative signs appears to be the most sensitive possible. The linear correlation coefficient between two sets of data $\{x_i\}$ and $\{y_i\}$ is

$$\rho(x,y) = \frac{\sum_i (x_i - \langle x \rangle) \sum_i (y_i - \langle y \rangle)}{[\sum_i (x_i - \langle x \rangle)^2 \sum_i (y_i - \langle y \rangle)^2]^{1/2}}. \tag{1}$$

If the amplitudes in each channel are Gaussian with zero mean,[12] then

$$\rho(\gamma_a^2, \gamma_b^2) = \rho^2(\gamma_a, \gamma_b). \tag{2}$$

Since this relation depends on joint moments of γ_a and γ_b in addition to the moments for each channel, it should be more sensitive than a test using only the widths in a single channel.

The only data to test Eq. (2) are from high-resolution proton inelastic scattering,[14] where interference between coherent channels in a reaction allows the determination of the relative sign of a pair of amplitudes. Details of the experiments are given in Ref. 14 and are only summarized here. For several even-even targets, angular distributions have been measured on resonance for inelastic scattering to the first excited state ($J^\pi=2^+$) and for the subsequent deexcitation γ ray to the ground state. For a given resonance, the analysis yields both the magnitudes and the relative signs of the reduced-width amplitudes in the inelastic channels (two for p-wave resonances, three for d-wave resonances). These data provide sufficient information to determine the correlation coefficients [Eq. (1)] of both widths and amplitudes for a set of compound nuclear resonances with the same spin and parity; each pair of channels for a given nucleus thus yields both an amplitude correlation and a width correlation. These are the first separate measurements of width and amplitude correlations. A sufficient number of resonances have been studied to consider a statistical analysis for three different J^π values ($\frac{3}{2}^-, \frac{3}{2}^+, \frac{5}{2}^+$) and four different compound nuclei (^{45}Sc, ^{49}V, ^{51}Mn, ^{57}Co). Although there are three decay channels for $\frac{3}{2}^+$ and $\frac{5}{2}^+$ states, one can consider the channels pairwise without loss of generality. Surprisingly large correlations are observed between the inelastic channels; these are presumed to indicate the presence of direct reactions.[14] It is the large values of the measured correlations (typically $\simeq 0.5$) which make practical a test of Eq. (2).

Some of these data seem to disagree with Eq. (2). Attention has focused on understanding the significance level of these discrepancies and the effects of experimental errors.[16] Harney[15] examined the effect of limited sample size on tests of Eq. (2) by calculating the uncertainty in the ratio $r^2(\gamma_a,\gamma_b)/r(\gamma_a^2,\gamma_b^2)$ due only to sample size (here r is the experimental value for the linear correlation coefficient, evaluated with finite averages from the data rather than ensemble averages as in the definition). He concluded that the uncertainty due to sample size alone made a conclusive test impossible with the individual data sets but that the average ratio for all data agreed with the expected value of 1.[17] However, another test[18] with the same approach, but considering the difference $r^2(\gamma_a,\gamma_b)-r(\gamma_a^2,\gamma_b^2)$ instead of the ratio, concluded that the finite sample size could not explain observed discrepancies and that the averaged data did not agree with the Gaussian assumption.

Because tests of Eq. (2) with a ratio[17] and a difference[18] lead to different results, we will bypass this difficulty by analyzing the data in a representation where the amplitude correlation r_a is zero (we denote width correlations by the subscript w and amplitude correlations by the subscript a). Therefore, in this "zero" representation only a single correlation coefficient is obtained for each pair of channels for a given J^π in a given nucleus. If the Gaussian assumption holds, then the value of r_w is also zero.

The procedure for changing representations has been discussed in Ref. 14. The same orthogonal transformation is applied to the amplitudes for each resonance in a data set. For $\frac{3}{2}^-$ resonances there are two amplitudes, and the general orthogonal transformation depends on a single rotation angle θ; the specific representation depends on the value of θ. For $l=2$ resonances, which have three amplitudes, the general transformation is described by Euler angles $\alpha\beta\gamma$ and involves a 3×3 matrix. Orthogonal transformations preserve the distributions in the sense that if the original amplitudes are Gaussian, the transformed amplitudes are also. We choose values of θ in the 2×2 case and $\alpha\beta\gamma$ in the 3×3 case such that for each data set, the amplitude correlations are all zero (simultaneously for the three pairwise correlations in the three-dimensional cases). The set of correlation coefficients for the resulting widths is unique. Although the labels a,b,c on the amplitudes do not have a definite meaning unless the exact transformation is specified (the transformed amplitudes can be permuted by different choices of $\alpha\beta\gamma$), we maintain the labels as a matter of bookkeeping.

The transformation to the zero representation was performed for six data sets involving d-wave resonances and three data sets involving p-wave resonances, yielding a total of 21 width correlations. The results are listed in Table I. In order to estimate the standard deviation σ for a given width correlation, we applied the bootstrap

TABLE I. Values of the width correlations in the zero representation.

Compound nucleus	J^π	n	γ_1	γ_2	r	σ	Signif. level (%)
^{45}Sc	$\frac{3}{2}^-$	37	γ_a	γ_b	−0.13	0.19	78
	$\frac{5}{2}^+$	53	γ_a	γ_b	−0.08	0.20	70
			γ_a	γ_c	0.21	0.12	95
			γ_b	γ_c	0.01	0.16	49
^{49}V	$\frac{3}{2}^-$	70	γ_a	γ_b	0.52	0.22	99.5
	$\frac{3}{2}^+$	30	γ_a	γ_b	−0.14	0.16	82
			γ_a	γ_c	0.33	0.21	89
			γ_b	γ_c	−0.09	0.10	95
	$\frac{5}{2}^+$	45	γ_a	γ_b	−0.24	0.07	99.0
			γ_a	γ_c	0.42	0.18	96
			γ_b	γ_c	−0.04	0.10	69
^{51}Mn	$\frac{3}{2}^-$	24	γ_a	γ_b	−0.13	0.20	75
	$\frac{5}{2}^+$	38	γ_a	γ_b	−0.05	0.07	83
			γ_a	γ_c	0.20	0.33	64
			γ_b	γ_c	0.03	0.23	66
^{57}Co	$\frac{3}{2}^+$	42	γ_a	γ_b	0.31	0.30	76
			γ_a	γ_c	0.11	0.14	81
			γ_b	γ_c	−0.13	0.08	95
	$\frac{5}{2}^+$	77	γ_a	γ_b	0.01	0.08	56
			γ_a	γ_c	0.45	0.13	>99.9
			γ_b	γ_c	0.07	0.10	79

method[19] to the widths in the zero representation. The problem is, given n data points and their linear correlation coefficient r, to determine the significance of r. The bootstrap method does this by sampling with replacement n data points from the original data set; any one datum point may be in the sample more than once or not at all. The correlation coefficient is then determined for this sample. The sampling procedure is repeated many times, and the resulting set of correlation coefficients is used as an approximation to the true distribution. Confidence levels can then be determined. This method does not assume any explicit distribution for the data and, therefore, seems ideally suited for the present analysis, in which the amplitude distribution is being tested.

We generated 25 000 samples for our bootstrap distributions; a typical distribution is shown in Fig. 1. Following the suggestion of Efron,[19] we take the value of σ to be $\frac{1}{2}$ the width of the central 68% of the distribution. We also list in Table I the significance level of each correlation as determined with the bootstrap; for a positive (negative) correlation coefficient, the significance level quoted is the fraction of correlations in the bootstrap sample which are positive (negative). We use the bootstrap to evaluate significance because it not only gives a direct measure of the effect of sample size but also reflects the experimental data much more than other methods. Three of the individual data sets disagree with the Gaussian assumption at a significance level of 99% and three more at the 95% level.

The central question is whether the combined data agree with the Gaussian assumption. In order to average the 21 different width correlations, one must choose appropriate weighting factors. Since the bootstrap method is a numerical and not an analytical technique, it is not clear what the weighting factor should be. Following the result of the usual analytic methods, we use the inverse variance $1/\sigma_i^2$ (the results are essentially the same for $1/\sigma_i$). The average width correlation is then

$$\bar{r}_w = \frac{\sum_i (r_i/\sigma_i^2)}{\sum_i (1/\sigma_i^2)}.$$

Combining the data in Table I yields $\bar{r}_w = -0.01$. To determine the significance of this value, we again apply a bootstrap, this time to $\bar{r}_w$, and obtain the distribution shown in Fig. 2. The significance level is only 56% ($\sigma = 0.04$), and the result is clearly consistent with $\bar{\rho}_w = 0$.

To summarize, for Gaussian distributions the width and amplitude correlations are simply related: $\rho_w = \rho_a^2$. Our data provide the only independent measurements of width and amplitude correlations. Previous analyses of these data considered the ratio r_a^2/r_w or the difference $r_a^2 - r_w$. The transformation of the data to a representation in which $r_a = 0$ leads to a value of r_w which can be compared directly with the predicted value of 0. The r_w's for 21 individual data sets are combined with a weighting of $1/\sigma_i^2$, where σ is one-half the width of the central 68% of the bootstrap distribution for each width correlation. The result is $\bar{r}_w = -0.01 \pm 0.04$. This provides the most direct and assumption-free confirmation that the reduced width amplitude distribution is multivariate Gaussian, as predicted by GOE, and further strengthens the proposed connection between random-matrix theory and quantum chaos.

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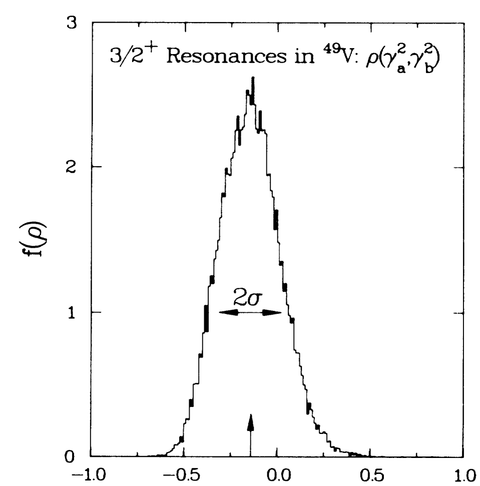

FIG. 1. Sample bootstrap probability distribution for a width correlation in the zero representation. The vertical arrow marks the value of the experimental correlation, while the horizontal one marks the central 68% of the distribution.

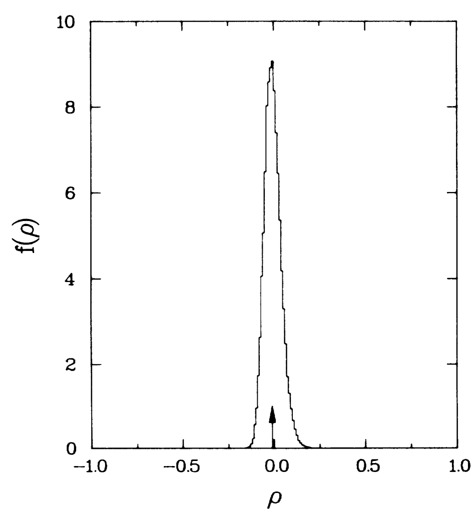

FIG. 2. Bootstrap distribution for the average width correlation determined from the 21 values of r_w. The value of $\bar{r}_w$ is −0.01.

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