Fermi-Liquid Theory of Interacting Disordered Systems and the Scaling Theory of the Metal-Insulator Transition

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The scaling variables in the theory of the interacting disordered system are interpreted in terms of the quasiparticle density of states and diffusion constant, and Fermi-liquid interaction parameters. In the absence of spin-flip or spin-orbit scattering, the scaling theory valid to first order in the disorder is interpreted and a general description of possible scaling scenarios is given.

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Understanding of the metal-insulator transition in interacting disordered systems has seen dramatic progress in the last three years. The breakthrough was provided by Finkelstein,[1-4] who identified the relevant scaling variables in the problem in terms of a mapping to a nonlinear σ model. A number of authors have noted the similarity of the results of Finkelstein's theory for physical quantities such as compressibility,[1] spin susceptibility,[2,5,6] and specific heat[7] to those expected in the Fermi-liquid theory. The correspondence between the couplings of the nonlinear σ model and the Fermi-liquid parameters was stressed in Ref. 7. A phenomenological transport theory for disordered interacting systems was written down by McMillan.[8] In this Letter we make this analogy precise by defining the notion of quasiparticles in a disordered medium. Finkelstein's theory can be understood in terms of Landau's basic assumptions, with one simplification—the interaction functions do not have complicated angular dependence—and a crucial complication—the Landau parameters are scale dependent.

Following Landau, we assume that the low-lying excitations of the interacting disordered system consists of quasiparticles (QP) and quasiholes; i.e., the unaveraged one-particle Green's function for small frequencies is dominated by simple poles plus an analytic incoherent term $\phi_{\rm inc}$:

$$G(\mathbf{x},\mathbf{x}',\omega) = \sum_n \frac{a_n \phi_n^*(\mathbf{x})\phi_n(\mathbf{x}')}{\omega-(E_n-\mu)+i\,\mathrm{sgn}(\omega)/2\tau_n} + \phi_{\rm inc}. \quad (1)$$

In Eq. (1), a_n is the QP weight, $E_n-\mu$ is the QP energy measured with respect to the Fermi energy, τ_n is the QP lifetime, and $\phi_n(\mathbf{x})$ is the "QP wave function" which can be defined formally as the basis which diagonalizes $G(\mathbf{x},\mathbf{x}',\omega)$ for $\omega\to 0$. Fleishman and Anderson[9] outlined a Fermi-liquid approach to the insulating phase, and gave general arguments to justify the diagonalizability of $G(\mathbf{x},\mathbf{x}',0)$. Their arguments apply in the metallic side as well.

To make further progress in the interpretation of Eq. (1), we must consider impurity-averaged quantities, which we denote by $\langle\ \rangle_i$. The simplest quantity is the QP density of states: $\rho_Q(E)=\langle V^{-1}\sum_n \delta(E-E_n)\rangle_i$, where V is the volume. We also introduce the single-particle density of states

$$\rho_1(\omega) = -\pi^{-1}V^{-1}\,\mathrm{Im}\int dx\, G(\mathbf{x},\mathbf{x},\omega)$$

which is measured in tunneling experiment. We have

$$\rho_1(E-\mu)=\langle V^{-1}\textstyle\sum_n a_n\delta(E-E_n)\rangle_i = a\rho_Q(E).$$

From now on we assume that the spectral weight is not correlated with the QP energy. We introduce a as the impurity average of a_n and we will be interested in E at the Fermi energy.

We next consider the function

$$L(\mathbf{q},\omega,\epsilon)=V^{-1}\int d^3x\, d^3x'\langle G_Q(\mathbf{x},\mathbf{x}',\epsilon+\omega)G_Q(\mathbf{x}',\mathbf{x},\epsilon)\rangle_i e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')},$$

where G_Q is the first term in Eq. (1). $L(q,\omega)$ is related to the Fourier transform of the probability of finding at $\mathbf{x}'$ a particle which was initially at $\mathbf{x}$. Assuming diffusive QP, i.e.,

$$\left|\int dx\,\phi_n(x)^* e^{-iqx}\phi_m(x)\right|^2 = D_Q q^2/[\pi\rho_Q V\{(E_n-E_m)^2+(D_Q q^2)^2\}],$$

we can show that[10]

$$L(\mathbf{q},\omega,\epsilon)=2\pi a^2\rho_Q\delta(\epsilon)\omega(D_Q q^2-i\omega+\tau_Q^{-1})^{-1}, \quad (2)$$

where D_Q is the QP diffusion constant and τ_Q is its average lifetime. The factor $\delta(\epsilon)\omega$ in Eq. (2) expresses the fact that $L=0$ unless $\epsilon+\omega>0$ and $\epsilon<0$. Note that the QP wave functions ϕ_n *are not the same* as the exact eigenstates of

noninteracting electrons in the same random potential, and D_Q will in general be very different from its value in the absence of interactions.

The quantity $L(q,\omega,\epsilon)$ can also be calculated in the field-theory formulation[1] or in perturbation theory.[5] This connection allows us to check the validity of the diffusive assumption implicit in Eq. (2), in perturbation theory in $1/k_f l$, and to calculate the relevant scaling quantities. In the perturbative approach, the incoherent part of G is absorbed into the definition of the static vertex function by taking advantage of the dependence on $\omega/D'q^2$ in a way analogous to clean Fermi-liquid theory. We obtain[5]

$$L(q,\omega,\epsilon)=2\pi\rho_0\zeta^2\delta(\epsilon)\omega/(D'q^2-iz\omega+\tau_{\rm ph}^{-1}),$$

where ρ_0 is the bare density of states, ζ is the renormalization of the single-particle density of states so that $\rho_1=\zeta\rho_0$, D' is the renormalized diffusion constant, z is the frequency renormalization, and $\tau_{\rm ph}$ is the phase relaxation time. Comparison with Eq. (2) shows that we should identify $D_Q=D'/z$, $\rho_Q=z\rho_0$, $\tau_Q=z\tau_{\rm ph}$, and $a=\zeta/z$. The interpretation of z as the renormalization of the QP density of states implies that γ (the linear term in the specific heat) is renormalized by z, i.e., $\gamma/\gamma_0=z=\rho_Q/\rho_0$. This is in agreement with renormalized perturbative analysis.[7]

Next the theory proceeds to analyze response function in terms of skeleton diagrams involving L and the short-range part of the static interaction amplitudes $\tilde{\Gamma}$. The only difference from the ordered Fermi-liquid theory is that momentum is not a conserved quantity, so that $\tilde{\Gamma}$ is characterized only by spin variables and is parametrized by singlet and triplet amplitudes $\tilde{\Gamma}_s$ and $\tilde{\Gamma}_t$ in the particle-hole channel. $\tilde{\Gamma}_s$ and $\tilde{\Gamma}_t$ are often parametrized as $\tilde{\Gamma}_s=\tilde{\Gamma}_1-\tilde{\Gamma}_2/2$ and $\tilde{\Gamma}_t=-\tilde{\Gamma}_2/2$. Just as in ordinary Fermi-liquid theory, we take advantage of the singular dependence of L on the ratio $D_Q q^2/\omega$ to write physical response functions in terms of scattering amplitudes on the Fermi surface. One then uses conservation laws (i.e., Ward identities) to derive the relations[5,6] $(dn/d\mu)/\rho_0=(\rho_Q/\rho_0)(1-2\gamma_s)$ and

$$\chi/\chi_0=(\rho_Q/\rho_0)(1-\gamma_2)=(\rho_Q/\rho_0)(1-2\gamma_t),$$

where χ is the spin susceptibility. This is the Fermi-liquid way of writing response functions as a product of density of states and interaction correction. We define $\gamma_i=\rho_0\zeta^2\Gamma_i/z$, $i=1,2,s,t$, and it is clear that $2\gamma_s$ and $2\gamma_t$ play the role of the Landau parameters A_0^s and A_0^a in the Fermi-liquid theory. The analogy with ordered Fermi-liquid theory is even more apparent if we rewrite $\gamma_i=a^2\rho_Q\tilde{\Gamma}_i$.

Finally the charge-diffusion constant $D\equiv\sigma/e^2(dn/d\mu)$, where σ is the conductivity, and the spin-diffusion constant D_S are given by[1,2,6] $D=D'\rho_0/(dn/d\mu)=D_Q/(1-2\gamma_s)$, $D_S=D'\chi_0/\chi=D_Q/(1+\gamma_2)$. We can also show that the thermal diffusion constant $D_T=D_Q$ without any correction due to the Landau parameters. These are the expected relations in a phenomenological Fermi-liquid description.

The parameters z, γ_1, γ_2, together with the disorder parameter $t=\epsilon_d\Lambda^{d-2}/\sigma$ (where ϵ_d is a numerical constant and $\Lambda\approx l^{-1}$ is the momentum upper cutoff), are the scaling parameters in the field theory[1] and they are all directly related to physically observables $dn/d\mu$, χ, D, D_S, and D_T. The presence of infrared divergences in the theory for $d=2$ requires a renormalization of the scaling parameters. Fluctuations with momentum $\lambda\Lambda<q<\Lambda$ and all frequencies are integrated out at each stage, and scaling equations for t, γ_i, and z are derived.[1-6] They fall naturally into different universality classes, depending on the presence of spin-flip or spin-orbit scattering or high magnetic field, and have been analyzed in the literature.[1-6,11] Here we interpret the nature of these transitions in terms of the QP picture. *A priori* a metal-insulator transition could be caused by either a vanishing D_Q or ρ_Q, or both. We shall show that all these scenarios are realized.

(a) *The general case.*[1-7]—In the absence of symmetry-breaking fields, the scaling equations for $d=2+\epsilon$ to first order in t and all orders in γ_2 are

$$\frac{dt}{d\xi}=-\frac{\epsilon t}{2}+t^2\left[4-3\frac{1+\gamma_2}{\gamma_2}\ln(1+\gamma_2)\right], \tag{3}$$

$$d\gamma_2/d\xi=\tfrac{1}{2}t(1+\gamma_2)^2, \tag{4}$$

where $\xi=-\ln\lambda^2$. There exists a stable fixed point at $\gamma_2^*=\infty$, $z=\infty$, and $t^*=0$, such that $\gamma_2^* t^*=\epsilon$. Near this fixed point an approximate solution is $\gamma_2^{-1}(\xi)=(\gamma_{20}^{-1}-t_0/\epsilon)+(t_0/\epsilon)e^{-\epsilon\xi/2}$, where γ_{20} and t_0 are the initial values. There is a critical line in the (γ_{20},t_0) plane (given by the condition $\gamma_{20}^{-1}-t_0/\epsilon=0$ for $\gamma_{20}\gg 1$) along which γ_2 flows to infinity. On one side of this line we have metallic behavior. The flows on the metal side of the transition end in a Fermi-liquid fixed line ($t=0$, γ_2 arbitrary) which represents all the possible Fermi-liquid fixed points (characterized by different values of the Fermi-liquid parameter γ_2). χ diverges as the critical line is approached. On the insulating side of the transition (when $\gamma_{20}^{-1}-t_0/\epsilon<0$), γ_2 diverges at a finite scale. At the same time, z diverges while $t(\lambda)$ and therefore $\sigma(\lambda)$ remain finite. Thus it was not clear whether this is a metal-insulator transition at all.

The Fermi-liquid theory we developed now permits a clear physical interpretation of the above results. The first observation is that as the transition is approached from the metallic side, D remains constant, but D_Q scales to zero since $z\to\infty$. Thus the transition corresponds to the localization of the QP and is a metal-insulator transition. At the transition, $t(\lambda)\sim\lambda^{\epsilon}$ so that $\sigma(\lambda)=\lambda^{-\epsilon}t(\lambda)$ scales to a constant. Since the QP's are localized we expect σ to be zero on the insulating side so that the zero-temperature conductivity jumps discontinuously at the

transition, i.e., we have a metal-insulator transition with a minimum metallic conductivity.[12] In terms of QP's, this surprising result can be understood by our writing $\sigma = e^2 \rho_Q D_Q$ and noticing that $z \approx \lambda^{-3\epsilon}$ and the divergence in ρ_Q compensates for the vanishing $D_Q \approx z^{-1} \approx \lambda^{3\epsilon}$. On the insulating side, we already mentioned that γ_2 and z diverge at $\lambda = \lambda_c$. It is now natural to interpret λ_c as the inverse localization length. We find that $\lambda_c^{-1} \sim |n - n_c|^{-\nu}$ with $\nu = 1/\epsilon$.

The tunneling density of states ρ_1 increases with decreasing energy scale but reaches a finite value

$$\rho_1 = \rho_0 \exp\{-\tfrac{1}{2} \int d(\xi) t(\xi) [\xi - 3 \ln(\gamma_2(\xi))]\}$$

at the transition. The QP weight $a = \zeta/z \approx \lambda^{3\epsilon}$ vanishes at the transition. We note that as far as the magnetic properties are concerned, one expects that on the localized side, the formation of local moment must lead to infinite susceptibility at zero temperature. In the absence of spin-flip scattering, this divergence must be reflected in the metallic side, as well. In the Fermi-liquid theory, the divergence of χ is driven by a simultaneous divergence of $\rho_Q/\rho_0 = z$ and γ_2. Thus our result is different from the Brinkman-Rice picture, where the χ divergence is entirely due to an infinite density of states. It is gratifying to see that the scaling theory produces a divergent χ even in the Fermi-liquid regime, and the vanishing of the spectral weight a may be interpreted as a gradual conversion of the elementary excitation from density fluctuation to spin fluctuation.

Finally it is necessary to check that the basic requirement, that QP excitations are well defined, should be obeyed, i.e., $\tau_Q^{-1} \ll kT$. It was found earlier that $\tau_{\rm ph}^{-1}$ is enhanced near the transition.[13] We find that near the transition, when $\gamma_2 \gg 1$,

$$\tau_Q^{-1}/T \approx 3\pi (t^* \gamma_2^*/\epsilon)[(T\tau)^{\epsilon/2} - (T\tau_Q)^{\epsilon/2}].$$

Expanding in ϵ, we find $\tau_Q^{-1}/T \approx 3\pi\epsilon$ up to logarithmic accuracy, so that in principle, the QP's are marginally well defined. Note that the appearance of z in τ_Q is crucial, since $\tau_{\rm ph}^{-1}$ itself becomes greater than kT in the ϵ expansion.

It is useful to discuss these results in the framework of a general scaling theory with the parameters t, γ_2, and z.[14] Let us focus on the density-density correlation function and parametrize it as

$$(dn/d\mu) D(q,\omega) q^2 / [D(q,\omega) q^2 - i\omega]$$

and introduce the conductivity $\sigma(q,\omega) = (dn/d\mu) \times D(q,\omega)$; then

$$\sigma(q,\omega,t,\gamma_2) = \lambda^{d-2} \sigma(q/\lambda, \omega/\lambda^{d+x}, t(\lambda), \gamma_2(\lambda)), \quad (5)$$

where the exponent x is defined by $z = \lambda^{-x}$. We have assumed that the first-order in t result that $dn/d\mu$ is not renormalized[1] is true in general. Consequently, the bare dimension λ^{d-2} appears in Eq. (5). One proceeds to expand about the fixed point t^*, γ_2^* and there are two possibilities. (i) t^* and γ_2^* are finite (which turns out to be the case in the presence of magnetic impurities, spin-orbit scattering, or a magnetic field), and we immediately obtain the result $\sigma \sim t^* \xi^{2-d}$. (ii) $t^* = 0$ and $\gamma_2^* = \infty$ so that some combination $f(\gamma_2^*)\, t^* = \text{const}$. In this case, t is a dangerously irrelevant variable; we can no longer set $t = t^*$ in Eq. (5). Instead, we introduce a new exponent θ by $t(\lambda) = \lambda^\theta$ and note that on the length scale ξ, σ is linear in t^{-1}, so that we obtain

$$\sigma(q,\omega) = \xi^{2-d+\theta} F(q\xi, \omega \xi^{d+x}), \quad (6)$$

where F is a dimensionless scaling function which goes to a constant when its arguments are zero. Thus the conductivity exponent μ (defined by $\sigma = |n - n_c|^\mu$) is given by $\mu = (d - 2 - \theta)\nu$. Similar considerations on the insulating side show that the dielectric constant E diverges as[14] $E = \xi^{2+x+\theta}$. We note that the solution of the first-order scaling equations (3) and (4) yields $\theta = d - 2$ and a minimum metallic conductivity results. This is because the second term in Eq. (3) is negligible near the fixed point. However, it is entirely possible that higher-order terms in Eqs. (3) and (4) will change the picture. For example, if the next-order term in Eq. (3) is $at^3\gamma_2^2$, this will lead to $\theta = d - 2 - 6a\epsilon^2$ so that the conductivity will vanish as a power law again. Worse yet, if the higher-order terms contain high powers of γ_2 (e.g., $t^3\gamma_2^3$), we will not have a systematic expansion in ϵ. Without knowing the functional form of the higher-order terms, we do not know which if these scenarios actually occurs.

(b) *Strong magnetic field,*[3,5,7] *magnetic impurities,*[3,5,7] *and the spin-orbit case.*[1,4,7]—All these cases belong to possibility (i), i.e., t^* is finite, and the metal-insulator transition has been described in detail earlier. We have checked that $\tau_Q^{-1}/T \ll 1$ is always obeyed in $2 + \epsilon$ dimensions and that the QP residue vanishes at the transition. In the case of strong magnetic field, ρ_Q is finite at the transition while D_Q vanishes. For magnetic impurities, both ρ_Q and D_Q vanish as $\lambda^{\epsilon/2}$. Interestingly, in the spin-orbit case, D_Q is finite at the transition while ρ_Q vanishes, so that this transition should not be considered a localization transition at all.

Finally, we briefly discuss the experimental situation. For alloys such as $Ge_{1-x}Au_x$ or compensated doped semiconductors, the conductivity appears to satisfy $\sigma \sim |x - x_c|^\mu$, where μ is of order unity.[15] These cases appear to belong to the class of strong spin-flip or spin-orbit scattering, and are in qualitative agreement with the theory. The one exception is uncompensated Si:P, where μ is found to be approximately $\frac{1}{2}$.[16] It is known experimentally that spin-flip and spin-orbits rates are much less than kT up to $n/n_c \approx 1.06$, with $kT \approx 30$ mK, so that this system should belong to the general case.[17] In the general case, there are two possible scenarios: (a) The solution is characterized by a minimum metallic

conductivity,[12] in which case we suggest that for n very near n_c, the spin-flip or spin-orbit rate is enhanced so that we crossover to the respective universality class, giving rise to an apparent exponent of $\mu = \frac{1}{2}$. This interpretation, while speculative at this point, can be tested by one's obtaining experimental measurements of the spin-flip or spin-orbit rate for n very near n_c and by making similar tests on compensated samples. (b) The conductivity vanishes as a power law given by Eq. (6). In this case, measurements of σ, E, and the specific heat will determine the three exponents ν, θ, and x, and the consistency of this picture can be checked from the temperature or frequency dependence of σ at the transition.[14] In addition, the rigorous bound $\nu > 2/d$ can be tested.[18] The choice between (a) and (b) awaits further theoretical and experimental work. It is also possible that the Fermi-liquid theory breaks down before we reach the metal-insulator transition because of the formation of local magnetic moments in the metal. However, to discuss this possibility we need a theory of the Kondo effect in a strongly disordered metal which is not available at present.

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[1]A. M. Finkelstein, Pis'ma Zh. Eksp. Teor. Fiz. **84**, 168 (1983) [Sov. Phys. JETP **57**, 97 (1983)].

[2]A. M. Finkelstein, Z. Phys. B **56**, 189–196 (1984).

[3]A. M. Finkelstein, Zh. Eksp. Teor. Fiz. **86**, 367 (1984) [Sov. Phys. JETP **59**, 212 (1984)].

[4]A. M. Finkelstein, Pis'ma Zh. Eksp. Teor. Fiz. **40**, 63 (1984) [JETP Lett. **40**, 796 (1984)].

[5]C. Castellani, C. DiCastro, P. A. Lee, and M. Ma, Phys. Rev. B **30**, 527 (1984).

[6]C. Castellani, C. DiCastro, P. A. Lee, M. Ma, S. Sorella, and E. Tabet, Phys. Rev. B **30**, 1596 (1984), and **33**, 6169 (1986).

[7]C. Castellani and C. DiCastro, to be published; C. Castellani and C. DiCastro, in *Localization and Metal Insulator Transition,* edited by H. Fritzche and D. Adler (Plenum, New York, 1985).

[8]W. McMillan, Phys. Rev. B **31**, 2750 (1985).

[9]L. Fleishman and P. W. Anderson, Phys. Rev. B **21**, 2366 (1980).

[10]C. Castellani, C. DiCastro, G. Forgacs, and S. Sorella, Solid State Commun. **52**, 261 (1984).

[11]E. Abrahams, P. W. Anderson, P. A. Lee, and T. V. Ramakrishnan, Phys. Rev. B **24**, 6783 (1982).

[12]N. F. Mott, Philos. Mag. **26**, 1015 (1972).

[13]C. Castellani, C. DiCastro, G. Kotliar, and P. A. Lee, Phys. Rev. Lett. **56**, 1179 (1986).

[14]Details of the general scaling theory will be given elsewhere.

[15]W. L. McMillan and J. Mochel, Phys. Rev. Lett. **46**, 556 (1981).

[16]T. F. Rosenbaum, R. F. Milligan, M. A. Paalanen, G. A. Thomas, R. N. Bhatt, and W. Lin, Phys. Rev. B **27**, 7509 (1983).

[17]M. Paalanen, S. Sachdev, R. N. Bhatt, and A. Ruckenstein, to be published.

[18]J. Chayes, L. Chayes, D. S. Fisher, and T. Spencer, unpublished.