PHYSICAL REVIEW LETTERS

VOLUME 59 | 28 DECEMBER 1987 | NUMBER 26

Exact Lyapunov Dimension of the Universal Attractor for the Complex Ginzburg-Landau Equation

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(Received 16 September 1987)

We present an exact analytic computation of the Lyapunov dimension of the universal attractor of the complex Ginzburg-Landau partial differential equation for a finite range of its parameter values. We obtain upper bounds on the attractor's dimension when the parameters do not permit an exact evaluation by our methods. The exact Lyapunov dimension agrees with an estimate of the number of degrees of freedom based on a simple linear stability analysis and mode-counting argument.

PACS numbers: 05.45.+b, 03.40.−t, 46.20.+e, 47.25.−c

The complex Ginzburg-Landau equation (CGLE),

$$A_t = \alpha A + \beta A_{xx} + \gamma A\,|A|^2, \qquad A(x,t), \alpha, \beta, \gamma \text{ complex}, \tag{1}$$

is a generic amplitude equation on long space and time scales close to the critical point of instability in a variety of problems, particularly in fluid dynamics[1] and in chemical turbulence.[2] Without loss of generality the linear driving coefficient α may be taken real. The coefficient α usually derives from how far the original problem is above criticality and is thus naturally taken as a bifurcation or control parameter. Various papers have investigated pattern formation, low-dimensional chaos, and coherent structures[3] in the CGLE in one spatial dimension with periodic boundary conditions ($x \in [0,L]$) in the parameter regime $\alpha > 0$, $\beta_r > 0$, and $\gamma_r < 0$. This is the regime which will concern us for the rest of this Letter.

The six parameters in the problem (L, α, β_r, β_i, γ_r, and γ_i) are not all independent. We may define the new dimensionless coordinates x', t', and the new variable A' by

$$x' = x/L \in [0,1], \quad t' = (\beta_r/L^2)t, \qquad A' = L(-\gamma_r/\beta_r)^{1/2}A, \tag{2}$$

and the dimensionless parameters

$$R = \alpha L^2/\beta_r, \quad \nu = \beta_i/\beta_r, \quad \mu = \gamma_i/\gamma_r, \tag{3}$$

and rewrite the CGLE (dropping the primes)

$$A_t = RA + (1+i\nu)A_{xx} - (1+i\mu)\,|A|\,A^2, \qquad x \in [0,1]. \tag{4}$$

The new bifurcation parameter R, playing the role of an effective Reynolds or Rayleigh number, is the ratio of the long-wavelength driving, or destabilizing, rate (α) to the long-wavelength damping, or restoring, rate (β_r/L^2).

The finite dimensionality of (possibly strange) attractors of *a priori* infinite-dimensional dissipative systems is a topic which has received much attention as it provides a link between chaos in finite-dimensional dynamical systems and turbulence in continuum systems. Finite-dimensional attractors have been rigorously established for an increasingly large number of partial differential equations, including the two-dimensional Navier-Stokes equations, the Kuramoto-Sivashinsky equation, and a family of reaction-diffusion equations.[4] In all these cases finite upper bounds on the dimensions have been obtained, while the exact dimensions have eluded computation.

The universal attractor, X, for the CGLE in the Hilbert space $L^2[0,1]$ is the largest set both invariant under the dynamics and bounded under the time-reversed dynamics.[5] In this Letter we compute the exact Lyapunov dimension of X as a function of R when $|\mu| \le \sqrt{3}$ [see Eq. (4)]. The Lyapunov dimension, d_L, defined by the Kaplan-Yorke formula[6] with the global Lyapunov exponents, is an upper bound on the Hausdorff dimension of X.[7] We obtain finite upper bounds on d_L as a function of R when $|\mu| > \sqrt{3}$.

All solutions of the CGLE are attracted to a bounded set in L^2 at a uniform exponential rate, uniformly in the initial condition. The time-dependent L^2 norm ($\|A\|_2^2 = \int_0^1 dx\, A^* A$) of any solution satisfies

$$\|A\|_2^2 \le R/[1-e^{-2Rt}], \tag{5}$$

independent of the initial condition. Similarly, the L^2 norm of the derivative (the H^1 norm) satisfies

$$\lim_{t\to\infty} \|A_x\|_2^2 \le \delta^2 R^3 \{1+[1+(1+\delta)/\delta^2 R]^{1/2}\}^2, \tag{6}$$

where $\delta = \max\{0, -2+|1+i\mu|\}$, and this limit is approached at a uniform exponential rate independent of the initial conditions. A detailed derivation of these estimates is presented by Doering *et al.*[8] The upshot of these considerations is that X lies in the ball of radius R in L^2. Additionally, $X \in L^\infty$ (the set of bounded functions) since the asymptotic L^2 and H^1 norms above imply that

$$\sup_X \|A\|_\infty^2 \le R + 2\delta R^2 \{1+[1+(1+\delta)/\delta^2 R]^{1/2}\}. \tag{7a}$$

In the cases where $v=0$ the CGLE admits a maximum principle and an improved upper bound on the L^∞ norm of the solutions on X may be derived[8]:

$$\sup_X \|A\|_\infty^2 \le R, \quad v=0. \tag{7b}$$

To compute the global Lyapunov exponents we consider the linearized flow along a trajectory in X. For a solution $A(t) \in X$ (with the spatial coordinate suppressed) the linearized flow of the vector $\xi(t) \in L^2$, along $A(t)$, is defined by

$$\partial_t \xi = R\xi + (1+iv)\xi_{xx} - 2(1+i\mu)|A|^2 \xi - (1+i\mu)A^2 \xi^* = F(t, A_0)\xi, \tag{8}$$

where $F(t,A_0)$ is shorthand for the generator of the linearized flow with the initial condition A_0 for the solution of the nonlinear flow. We will denote the solution $\xi(t)$ of Eq. (8) as $L(t,A_0)\xi$ where ξ is the initial condition. The sum of the first n global Lyapunov exponents[7] governs the largest exponential growth rates of n volumes according to

$$\lambda_1 + \cdots + \lambda_n := \limsup_{t\to\infty} t^{-1} \ln \left\{ \sup_{A_0 \in X} \sup_{\|\xi_i\| \le 1} \|L(t,A_0)\xi_1 \wedge \cdots \wedge L(t,A_0)\xi_n\| \right\}. \tag{9}$$

In the above the magnitudes of the volume elements are given by the norms on the corresponding spaces of n-forms on L^2. That is, for vectors $\xi_i, \zeta_j \in L^2$,

$$\|\xi_1 \wedge \cdots \wedge \xi_n\|^2 = \langle \xi_1 \wedge \cdots \wedge \xi_n, \xi_1 \wedge \cdots \wedge \xi_n \rangle, \quad \langle \xi_1 \wedge \cdots \wedge \xi_n, \zeta_1 \wedge \cdots \wedge \zeta_n \rangle = \det M, \quad \text{with } M_{ij} = \langle \xi_i, \zeta_j \rangle, \tag{10}$$

where $\langle A,B \rangle$ is the inner product in L^2 [in our situation, $\langle \xi, \zeta \rangle = \int_0^1 dx\, \xi^*(x)\zeta(x)$].

The Lyapunov dimension of X is defined by the following procedure. Consider the integer m such that

$$\lambda_1 + \cdots + \lambda_m \ge 0, \quad \text{but } \lambda_1 + \cdots + \lambda_{m+1} < 0. \tag{11}$$

Then d_L is computed from the Kaplan-Yorke formula[6]

$$d_L := m + (\lambda_1 + \cdots + \lambda_m)/|\lambda_{m+1}|. \tag{12}$$

The fundamental theorem of Constantin and Foias[7] asserts that d_L is an upper bound on the Hausdorff dimension of X when the global Lyapunov exponents are utilized.

The time derivative of the n-volume spanned by $\xi_1(t), \ldots, \xi_n(t)$ is given by

$$(d/dt)\|L(t,A_0)\xi_1 \wedge \cdots \wedge L(t,A_0)\xi_n\|^2 = 2\|L(t,A_0)\xi_1 \wedge \cdots \wedge L(t,A_0)\xi_n\|^2 \operatorname{Re}\{\operatorname{Tr}[F(t,A_0) \circ P_n(t)]\}, \tag{13}$$

where $P_n(t)$ denotes the time-dependent projection of L^2 onto the span of $\xi_1(t), \ldots, \xi_n(t)$ and $\operatorname{Tr}[F(t,A_0) \circ P_n(t)]$ denotes the trace of the finite-rank operator $F(t,A_0) \circ P_n(t)$. From Eq. (9), the sum of the first n global Lyapunov exponents may be expressed as

$$\lambda_1 + \cdots + \lambda_n = \limsup_{t\to\infty} t^{-1} \ln \left\{ \sup_{A_0 \in X} \sup_{\|\xi_i\| \le 1} \exp\left[\operatorname{Re}\left(\int_0^t ds \operatorname{Tr}[F(s,A_0) \circ P_n(s)] \right) \right] \right\}. \tag{14}$$

Although the ξ_i's are not explicitly present in Eq. (14) above, they enter the formula via the time-dependent projection $P_n(s)$.

A lower bound on the sum of the first n global Lyapunov exponents is immediately obtained by our noting that the initial condition $A_0 = 0$ is contained in the universal attractor, corresponding to the nonlinear solution $A(t) \equiv 0$. Thus,

$$\sup_{A_0 \in X} \sup_{\|\xi_i\| \le 1} \exp\left[\operatorname{Re}\left(\int_0^t ds [\operatorname{Tr} F(s,A_0) \circ P_n(s)] \right) \right] \ge \exp\left[\operatorname{Re}\left(\int_0^t ds [\operatorname{Tr} F(s,0) \circ P_n'(s)] \right) \right], \tag{15}$$

where P_n' is the projection onto the first n Fourier coefficients $\phi_j(x)=\exp\{ik_jx\}$, in the order $k_1=0$, $k_2=2\pi$, $k_3=-2\pi$, $k_4=3\pi$, $k_5=-3\pi$, etc. The trace in the last term of Eq. (15) above is easily evaluated:

$$\mathrm{Tr}[F(t,0)\circ P_n'] = \sum_{j=1}^{n} \{R-(1+i\nu)k_j^2\}. \tag{16}$$

Hence we have the lower bound

$$\lambda_1+\cdots+\lambda_n \geq \sum_{j=1}^{n} \{R-k_j^2\}. \tag{17}$$

This lower bound is independent of the imaginary diffusion, ν, and the imaginary part of the nonlinear coupling, μ.

Upper bounds on the sum of the first n global Lyapunov exponents are obtained by our bounding the real part of the trace in Eq. (14) from above. Let ψ_j be a set of orthonormal vectors spanning $P_n(t)L^2$. Then

$$\mathrm{Re}\{\mathrm{Tr}[F(t,A_0)\circ P_n(t)]\} = \sum_{j=1}^{n} \{\langle\psi_j,(R+\partial_x^2)\psi_j\rangle - 2\langle\psi_j,|A|^2\psi_j\rangle - \mathrm{Re}[(1+i\mu)\langle\psi_j,A^2\psi_j^*\rangle]\}. \tag{18}$$

However, for any vector $\psi\in L^2$ and any $A_0\in X$,

$$-2\langle\psi,|A|^2\psi\rangle - \mathrm{Re}\{(1+i\mu)\langle\psi,A^2\psi^*\rangle\} = -2\int dx\,|A|^2|\psi|^2 - \mathrm{Re}\left\{(1+i\mu)\int dx\,A^2\psi^{*2}\right\} \leq \delta\|A\|_\infty^2\|\psi\|_2^2, \tag{19}$$

where $\delta=\max\{0,-2+|1+i\mu|\}$ as before, and $\|A\|_\infty^2$ is a uniform L^∞ bound on all solutions on X [Eqs. (7)]. Thus, utilizing Eq. (14) we obtain the upper bound

$$\lambda_1+\cdots+\lambda_n \leq \sum_{j=1}^{n} \{R+\delta\|A\|_\infty^2-k_j^2\}. \tag{20}$$

Note that $\delta=0$ when $|\mu|\leq\sqrt{3}$, so that the upper bound [Eq. (20)] coincides with the lower bound [Eq. (17)], thereby yielding the global Lyaponov exponents exactly. We remark that the computation of the Lyapunov exponents for $|\mu|\leq\sqrt{3}$ does not depend on the dimension of the space in which the CGLE is posed: The same formula holds for the CGLE when the spatial variable x lives in a bounded domain in R^d (with the spectrum of the d-dimensional Laplacian treated appropriately) provided that $A\equiv 0$ is an exact solution.

When $|\mu|\leq\sqrt{3}$, the Lyapunov exponents are

$$\lambda_n = R-(2\pi)^2[n/2], \tag{21}$$

where the bold square brackets indicate the integer part. The Lyapunov dimension is computed directly from the defining procedure, Eqs. (11) and (12). A plot of d_L versus the control parameter is given in Fig. 1. For $R>0$ it is a continuous curve with a discontinuous derivative at the points where d_L is an odd integer (i.e., d_L is not an analytic function of R). We may compute an analytic upper bound on the Lyapunov dimension which is exact at the points where d_L is an odd integer:

$$d_\mathrm{L} \leq 2(3R/4\pi^2+\tfrac{1}{4})^{1/2}. \tag{22}$$

This upper bound is also plotted in Fig. 1. The Lyapunov dimension is uniform in μ as well as ν in this parameter regime. Lower bounds on the Hausdorff dimension of X are easily obtained by our computing the dimension of the linearly unstable and neutral manifolds of the solution $A\equiv 0$. The linearization of the CGLE around this trivial solution yields a linear operator whose spectrum is easily determined to have the real parts $R-k_n^2$ [these are just the Lyapunov exponents in Eq. (21)]. Thus the trivial solution has $1+2[R^{1/2}/2\pi]$ unstable or neutral orthogonal directions and this serves as a lower bound on the Hausdorff dimension of X. This lower bound is also plotted in Fig. 1. The upper and lower bounds are both asymptotically proportional to

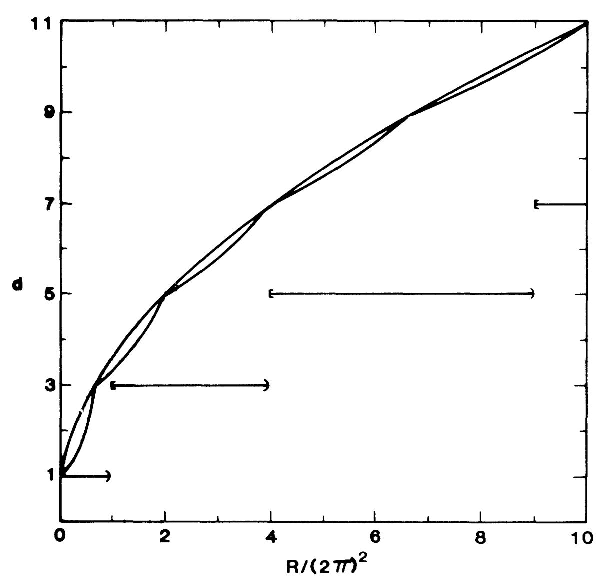

FIG. 1. Plot of the Lyapunov dimension vs effective Reynolds number for the one-dimensional complex Ginzburg-Landau equation with $|\mu|\leq\sqrt{3}$ (piecewise differentiable curve) and the upper bound [smooth curve from Eq. (22)]. The lower piecewise constant curve is the lower bound on the attractor's Hausdorff dimension obtained from a linear stability analysis of the trivial solution $A\equiv 0$.

$R^{1/2}/2\pi$, and their values for large R differ only by a factor of $\sqrt{3}$.

It is worthwhile to note that the $R^{1/2}/2\pi$ dependence of the dimension of X agrees with the intuitive notion of the "number of modes that fit into the box."[9] Expressed in terms of the original variables in Eq. (1), the attractor dimension is proportional to $R^{1/2}/2\pi = L_{\rm box}/L_{\rm diss}$ where $L_{\rm box}$ is the length of the interval and the "dissipation length," $L_{\rm diss} = 2\pi(\beta_r/\alpha)^{1/2}$, is the shortest wavelength excited by the linearized evolution. Excitations with wavelengths less than $L_{\rm diss}$ are damped by the linear part of the evolution operator. Simply stated, the result of our analysis is that when $|\mu| \le \sqrt{3}$ the largest Lyapunov exponents on X are those associated with the trivial solution $A=0$—for all values of the imaginary diffusion ν—in agreement with the more detailed linear stability analysis in Ref. 8.

When $|\mu| > \sqrt{3}$ the trivial solution is not necessarily the "most unstable" point on the attractor,[8] and we determine upper bounds on $d_{\rm L}$ rather than computing it exactly. If the sum of the first n Lyapunov exponents is less than or equal to 0, then $d_{\rm L} < n+1$. From the upper bound on the sum of the first n Lyapunov exponents [Eq. (20)] and the expressions for $\|A\|_\infty^2$ [Eqs. (7)], the upper bounds on $d_{\rm L}$ for large R may be summarized:

$$d_{\rm L} < (\sqrt{3}/\pi)|\mu| R + 3|\mu|^{1/2} R^{1/2}/2\pi + 2, \qquad |\mu| > \sqrt{3}, \quad \nu \text{ arbitrary}, \tag{23a}$$

$$d_{\rm L} < 2\sqrt{3}(1+|\mu|)^{1/2}(R^{1/2}/2\pi) + 2, \qquad \nu = 0, \quad \mu \text{ arbitrary}. \tag{23b}$$

We note that Eq. (23b) agrees with the lower bound while Eq. (23a) is significantly larger.

Other aspects of the finite-dimensional behavior of solutions to the one-dimensional CGLE are developed in Ref. 8. Notable among these is the fact that the CGLE also admits a finite-dimensional "inertial manifold." An inertial manifold is a smooth (Lipschitz) exponentially attracting invariant manifold which contains the universal attractor.[10] The partial differential equation, restricted to the inertial manifold, is equivalent to a finite number of ordinary differential equations. Since the inertial manifold is exponentially attracting we may assert that, modulo an exponentially decaying transient, the CGLE is in fact a finite-dimensional dynamical system.

This work was performed under the auspices of the U.S. Department of Energy under Contract No. W-7405-ENG-36P and the U.S. Office of Scientific Computing. After this work was completed we became aware of a parallel study[11] of low dimensionality in the CGLE. We thank J. Ghidaglia for bringing this to our attention.

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