Intensity Correlation Functions and Fluctuations in Light Scattered from a Random Medium

Michael J. Stephen and G. Cwilich
Serin Physics Laboratory, Rutgers University, Piscataway, New Jersey 08854
(Received 27 March 1987)

The propagation of light in a disordered medium with purely elastic scattering is considered. The intensity correlation functions for the diffusely reflected and transmitted light from a slab are shown to have long-range power-law behavior. This arises from the interference of the underlying wave fields of the diffusive modes of the light intensity. This interference also gives rise to anomalous fluctuations in the transmission coefficient analogous to conductance fluctuations in metals.

PACS numbers: 42.20.-y

There has been considerable interest recently in the propagation, multiple scattering, and localization of waves in disordered media. The effects of weak localization give rise to coherent backscattering which has recently been observed.[1-4] Large intensity fluctuations in light scattering are also seen.[3,4] In this Letter we discuss the intensity fluctuations and correlation functions of light scattered from a disordered medium. A large amount of literature on the subject of multiple scattering exists, and for recent reviews we refer to Ishimaru[5] and Goodman.[6] However, little systematic work on fluctuations and correlation functions in multiple scattering has been done. Shapiro[7] has recently considered the fluctuations in the light scattered from a point source in a disordered medium. His results correctly give the behavior of the correlation functions at short distances (less than a mean free path l) but not at large distances. This is discussed further below. Analogous problems to those considered here arise in the conductance and its fluctuations of electrons in disordered metals, and there has been considerable experimental[8] and theoretical[9-12] interest recently in this area. The relation of these results to those in this Letter are discussed below.

We consider an elastic scattering medium in the form of the slab of area A and thickness L, all dimensions being greater than l. The area is uniformly illuminated with monochromatic light of wave vector $k=\omega/c$ at normal incidence. We are interested in the light which is diffusely reflected or transmitted from the slab. We define cumulant correlation functions for reflection r and transmission t:

$$\begin{aligned} C^{(r)}(R) &= \langle |E(R,l)|^2 |E(0,l)|^2 \rangle_c, \\ C^{(t)}(R) &= \langle |E(R,L-l)|^2 |E(0,L-l)|^2 \rangle_c, \end{aligned} \tag{1}$$

where E is the field of the scattered light. We take this to be a scalar as only the scalar-type modes show diffusive behavior.[12] The plane of incidence of the light is $z=0$, and for reflection we evaluate the scattered field on the plane $z=l$ and for transmission on the plane $z=L-l$. This choice only affects the numerical value of the results. $\mathbf{R}$ is a point on the transverse surface $z=l$ or $L-l$. Light can only escape from the surfaces $z=0$, L of the slab and reflecting boundary conditions are imposed on the other surfaces.

Shapiro[7] has considered correlation functions such as (1) for the case of a point source of light in an infinite medium. His results are obtained by the assumption that the correlation functions factorize, e.g.,

$$C^{(r)}(R) = |\langle E(\mathbf{R},l) E^*(\mathbf{0},l)\rangle|^2$$

and in the above geometry give (for unit incident intensity)

$$\begin{aligned} C^{(r)}(R) &= C_r \left[\frac{\sin(kR)}{kR}\right]^2 e^{-R/l}, \\ C^{(t)}(R) &= C_t \left[\frac{l}{L}\right]^2 \left[\frac{\sin(kR)}{kR}\right]^2 e^{-R/l}, \end{aligned} \tag{2}$$

where the C are numerical constants of order 1. Equations (2) predict that the correlation functions decay in a mean free path and within this length exhibit a rapidly varying structure determined by the wavelength. These results are only correct for $R<l$. If we set $R=0$ in (2), we obtain the large intensity fluctuations discussed by Shapiro[7] and well known as the speckle patterns observed in scattering of light from random media.

The motion of the light in the disordered medium is diffusive as the scattering is purely elastic, and the long-range behavior of the correlations functions (1) is due to the interaction of diffusion modes in the scattering medium. The leading diagrams in an expansion in $(kl)^{-1}$ for the correlation functions are shown in Fig. 1. The correlation functions (1) involve four fields, and in these diagrams we have two diffusion modes which interfere in the medium because of the underlying wave field. This interference occurs in the bulk of the medium and, as diffusion is a long-range phenomenon, this effect leads to long-range correlations varying essentially as R^{-3}.[13] Similar diagrams arise in the electrical conductivity.[14] A straightforward calculation following previous meth-

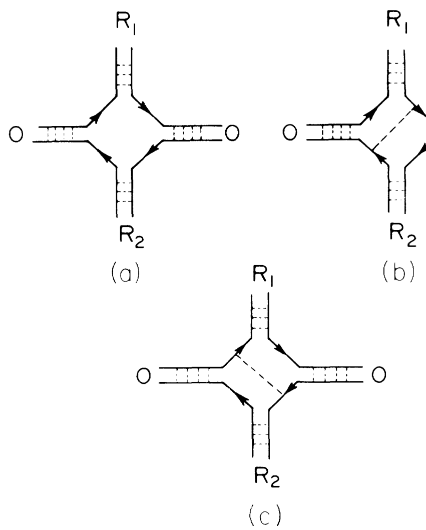

FIG. 1. Interaction of diffusion modes for an intensity correlation function $C(R_1 - R_2)$. The light is incident at 0. Lines with arrows directed from 0 to $\mathbf{R}_{1,2}$ are retarded E fields, and the reverse arrows indicate advanced E^* fields. The dotted lines indicate impurity scattering. Note that there is an important cancellation between (a) and (b) and (c) as discussed by Hikami (Ref. 13).

ods[12] gives

$$C^{(r)}(R) = C_r' \frac{27}{2k^2 l^2}\left(\frac{l}{R}\right)^3, \tag{3}$$

$$C^{(t)}(R) = \frac{27}{2k^2 l^2}\left(\frac{l}{L}\right)^3 \left[\frac{L}{R} + F_1(R/L)\right], \tag{4}$$

where C_r' is a numerical constant, and

$$F_1(R/L) = \frac{1}{2}\int_0^\infty dq\, J_0(qR/L)\left[\frac{\sinh(2q) - 2q}{\sinh^2 q} - 2\right] \tag{5}$$

where J_0 is a Bessel function. There are other diffusion interference diagrams in which the interference takes place near the surface of the sample. These diagrams contribute terms of the same order as in (3) in reflection but only higher-order effects to (4) in transmission.

The intensity correlation function (3) and (4) have a power-law form for $R > l$. They are of the same order of magnitude as (2) for $R = 1$. We have supposed that the transverse dimension of the slab $A^{1/2} \gg R$. In transmission the correlation functions depend on R/L and, for $R/L < 1$, $F(R/L) \simeq -1 + (R^2/4L^2)\zeta(3)$, where ζ is a Riemann zeta function, and thus $C^{(t)}$ varies slowly with R. For $R > L$, $C^{(t)}(R) \sim e^{-\pi R/L}$ and decreases exponentially. It should also be noted that the correlation functions are proportional to the small parameter $(kl)^{-2}$.

These results can be generalized to the case where there are two incident beams differing in frequency by $\Delta\omega$. We are interested in the correlation function of the scattered light intensity at ω and $\omega + \Delta\omega$:

$$C_{\Delta\omega}(R) = \langle |E_\omega(R,z)|^2 |E_{\omega+\Delta\omega}(0,z)|^2 \rangle_c, \tag{6}$$

where $z = l$ or $L - l$. Another length $\gamma^{-1} = (2cl/3\Delta\omega)^{1/2}$ now enters, and we suppose that this length is comparable to R and L and define dimensionless parameters $\beta_R = \gamma R$ and $\beta_L = \gamma L$. In the Shapiro approximation the correlation functions (2) decay exponentially with β_R and β_L. The correlation functions (3) and (4) now become

$$C_{\Delta\omega}^{(r)}(R) = \frac{27}{2k^2 l^2}\left(\frac{l}{R}\right)^3 [C_a' + \beta_R^2 F_2(\beta_R)], \tag{7}$$

where

$$F_2(\beta_R) = \frac{4}{\pi}\int_0^\infty dq\, q^2 K_0(q)/(q^2 + \beta_R^2),$$

where K_0 is a Bessel function.

In transmission,

$$C_{\Delta\omega}^{(t)}(R) = \begin{cases} C^{(t)}(R) + O(\beta_L^4), & \beta_L \ll 1, \\ \dfrac{27}{2k^2 l^2}\dfrac{l^3}{L^3 \beta_L}\displaystyle\int_0^\infty dq \frac{j_0(qR/L) q^3}{\sinh^2 q}, & \beta_L \gg 1, \end{cases} \tag{8}$$

where $C^{(t)}(R)$ is given in (4).

In reflection, the frequency difference does not have an important effect because in reflection it is not necessary that all diffusion paths be long. In transmission, the correlations decrease as β_L^{-1} for $\beta_L > 1$. For $\gamma l > 1$, the correlation function (6) goes to zero rapidly.

It is also possible to consider the correlation function of the diffusely scattered light emitted at different angles. We again consider the slab geometry in which monochromatic light is incident normally on the surface and determine the correlation function for the light emitted from a point on the surface in directions making angles $\pm\theta/2$ with respect to the normal (for either reflection or transmission). It is not difficult to show that this correlation function, $C^{(r,t)}(\theta)$ say, is proportional to the Fourier transform of the correlation functions (1) in the plane of reflection or transmission. Let

$$C^{(r,t)}(R) = \frac{1}{(2\pi)^2}\int d^2q\, e^{i\mathbf{q}\cdot\mathbf{R}} \hat{C}^{(r,t)}(q); \tag{9}$$

then $C^{(r,t)}(\theta) \sim \hat{C}^{(r,t)}(q)$ with momentum transfer $q = 2k\sin(\theta/2)$. For small angles we find

$$\hat{C}^{(r)}(q) = \frac{27\pi}{k^2}(C_a'' - qlC_a' + \dots), \quad \hat{C}^{(t)}(q) = \frac{27\pi}{k^2}\left(\frac{l}{L}\right)^2 \left[\frac{2}{3} + \frac{\sinh(2qL) - 2qL}{2ql\sinh^2(qL)} - ql + \dots\right]. \tag{10}$$

In the reflected case the correlations vary slowly with angle, while in the transmitted case there is an initial rapid decrease at $qL \sim 1$ followed by a similar slow decrease.

Finally we turn to a discussion of the reflection and transmission coefficients r and t, respectively, which we define for unit incident intensity by

$$r = \frac{1}{A}\int d^2R\,|E(R,l)|^2,$$
$$t = \frac{1}{A}\int d^2R\,|E(R,L-l)|^2. \qquad (11)$$

The averages of these quantities are $\langle r\rangle \sim 1$ and $\langle t\rangle \sim l/L$. Similar quantities are considered in the conductance of electrons in metals. An important difference is that in the present case the light incident on the slab excites a single incident channel. A quantity analogous to the average conductance is obtained by multiplying $\langle t\rangle$ by the number, Ak^2, of incident channels. The fluctuations in r and t follow from the Fourier transforms of (1). Thus from (9) and (10)

$$\langle r^2\rangle_c = \frac{1}{A}\hat{C}^{(r)}(0) = \frac{27\pi}{k^2l^2}\left(\frac{l^2}{A}\right)C_a'', \qquad (12)$$

$$\langle t^2\rangle_c = \frac{1}{A}\hat{C}^{(t)}(0) = \frac{18\pi}{k^2l^2}\left(\frac{l}{L}\right)\left(\frac{l^2}{A}\right). \qquad (13)$$

From the correlation functions obtained by Shapiro [Eqs. (2)], a similar result to (12) is obtained while, for transmission, a contribution smaller than (13) by a factor l/L is obtained. Equation (13) differs from the universal conductance fluctuations found for metals because only a single incident channel is excited in the optical case, but the implications are similar. The assumption of uncorrelated fluctuations[15] in each transmission channel leads to a result smaller than (13) by a factor l/L. Thus the interference of the diffusive modes leads to important correlations between the different transmission channels.

In conclusion, the transport of light energy in a disordered medium with purely elastic scattering is diffusive in nature. The underlying wave field gives rise to interference effects between diffusive modes. The long-range nature of diffusion leads to long-range correlations in the intensity correlation functions. It would be of interest to observe these correlation functions experimentally. The intensity fluctuations also lead to fluctuations in the reflection and transmission coefficients. These fluctuations are analogous to the universal conductance fluctuations in metals. The optical case considered here provides the opportunity to observe the long-range correlations responsible for reflection and transmission fluctuations.

This work was supported in part by the National Science Foundation under Grant No. DMR-84-05619.

[1]M. P. van Albada and A. Lagendijk, Phys. Rev. Lett. **55**, 2692 (1985).

[2]P. E. Wolf and G. Maret, Phys. Rev. Lett. **55**, 2696 (1985).

[3]S. Etemad, R. Thompson, and H. J. Andrejco, Phys. Rev. Lett. **57**, 575 (1986).

[4]M. Kaveh, M. Rosenbluh, I. Edrei, and I. Freund, Phys. Rev. Lett. **57**, 2049 (1986).

[5]A. Ishimaru, *Wave Propagation and Scattering in Random Media* (Academic, New York, 1978).

[6]J. W. Goodman, J. Opt. Soc. Am. **66**, 1145 (1976).

[7]B. Shapiro, Phys. Rev. Lett. **57**, 2168 (1986).

[8]R. A. Webb, S. Washburn, C. P. Umbach, and R. B. Laibowitz, in *Localization, Interaction and Transport Phenomena in Impure Metals,* edited by G. Bergmann, Y. Bruynseraede, and B. Kramer (Springer-Verlag, New York, 1985).

[9]B. L. Altshuler, Pis'ma Zh. Eksp. Teor. Fiz. **41**, 530 (1985) [JETP Lett. **41**, 648 (1985)].

[10]P. A. Lee and A. D. Stone, Phys. Rev. Lett. **55**, 1622 (1985).

[11]B. L. Altshuler and D. E. Khmelnitskii, Pis'ma Zh. Eksp. Teor. Fiz. **42**, 291 (1985) [(JETP Lett. **42**, 359 (1986)].

[12]M. J. Stephen and G. Cwilich, Phys. Rev. B **34**, 7564 (1986).

[13]This dependence follows from scaling. The diagrams of Fig. 1 give integrals of the form

$$\int d^3R \frac{1}{R^2}(\nabla_1\cdot\nabla_2)\frac{1}{|R-R_1|}\frac{1}{|R-R_2|} \propto \frac{1}{R_1^3}f(R_2/R_1).$$

The four factors $1/R$, $1/R$, $1/|R-R_1|$, and $1/|R-R_2|$ come from the four diffusion ladders and the gradient operators from the expansion of the Hikami box.

[14]S. Hikami, Phys. Rev. B **24**, 2671 (1981).

[15]P. A. Lee, A. D. Stone, and H. Fukuyama, to be published.