PHYSICAL REVIEW
LETTERS

VOLUME 59 | 21 DECEMBER 1987 | NUMBER 25

# Davey-Stewartson I System: A Quantum (2+1)-Dimensional Integrable System

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(Received 3 June 1987)

The Davey-Stewartson I equation is a nonlinear evolution equation originally derived in the context of multidimensional, weakly nonlinear water waves. It has recently been exactly solved by the classical inverse-scattering method for localized potentials, and also possesses nonlocal soliton solutions. We have calculated Poisson-bracket relations for elements of the scattering matrix, as well as corresponding quantum commutation relations. Commutation relations are found that are a (2+1)-dimensional generalization of a Yang-Baxter algebra.

PACS numbers: 03.70.+k, 03.65.−w, 11.10.Lm

Exactly solvable systems have played a significant role in our understanding of nonperturbative phenomena in physics. Many quantum field theories in 1+1 dimensions have been found to be integrable, allowing the calculation of exact $S$ matrices and physical spectra. The Ising model and other exactly solvable models of two-dimensional statistical mechanics have helped to provide a basis for modern scaling theory. Moreover, some of the more interesting mathematics occurring in quantum string theories, including loop spaces and Kac-Moody-Virasoro algebras, also appears in integrable systems.

Associated with every known integrable quantum system in two dimensions (or equivalently one space+one time dimension) is a solution of the so-called Yang-Baxter (YB) equations,[1,2] and the existence of a corresponding "Yang-Baxter algebra." The YB equations arise in various contexts, and have come to be regarded as the criterion for exact quantum integrability.

It is certainly of interest to extend the study of quantum integrability to more dimensions. Progress was made in this direction when Zamolodchikov considered the scattering of "straight strings" in a plane, and wrote down a 3D generalization of the YB equations, called the tetrahedron equations, as well as a conjectured solution.[3] Baxter was able to verify this solution and to calculate exactly the free energy of an equivalent classical statistical mechanical model.[4] However, the physical interpretation of the Zamolodchikov-Baxter solution is somewhat problematic, and the tetrahedron equations are so complex that little progress has been made in finding other solutions.

An alternative approach to the search for new quantum integrable systems in more dimensions is to exploit our knowledge of existing integrable classical systems. There now exist a number of nonlinear evolution equations in 2+1 dimensions [(2+1)D] which are solvable[5] by the classical inverse-scattering transform (CIST).[6] According to our experience in (1+1)D, each of these classical systems should have a corresponding quantum analog which is exactly integrable. In this Letter we consider the quantum analog of such a classical system, known as Davey-Stewartson.

The Davey-Stewartson (DS)[7] equation is a nonlinear partial differential equation in (2+1)D, originally formulated as a model to describe the evolution of multidimensional, weakly nonlinear water waves. Depending on the choice of the parameters in the equation, it admits two types of soliton solutions, localized lumplike solitons and nonlocalized straight-line-like solitons. Classically, the asymptotic scattering of the lump solitons is trivial, but the line solitons experience a nontrivial phase shift. For our purposes DS is an obvious choice because it is one of the simplest of the known more-dimensional integrable classical systems. It reduces in the (1+1)D limit to the well-known nonlinear Schrödinger (NLS) system, whose quantum version, the $\delta$-function gas model,[8] or quantum NLS model,[9] is one of the best understood of the quantum integrable systems.

We have calculated various classical Poisson-bracket relations between elements of the scattering matrix of the underlying linear problem for DS I, enabling us to identify explicitly the classical action-angle variables. We have also calculated certain corresponding quantum commutation relations and find them to be a (2+1)D generalization of a Yang-Baxter algebra.

We first discuss the classical case. We will be concerned with the hyperbolic version of the DS equation, a nonlinear partial differential equation for a complex-valued function $q=q(x,y,t)$,

$$i\frac{\partial q}{\partial t}=-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)q+iA_1q-iqA_2, \tag{1}$$

where

$$\begin{aligned}\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)A_1&=\frac{-i}{2}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)(qr),\\ \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)A_2&=\frac{i}{2}\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)(rq),\end{aligned} \tag{2}$$

with $r=\pm q^*$ ($q^*$ denoting the complex conjugate of $q$). This time-evolution equation for $q$ can be generated by a nonlocal Hamiltonian (which will depend on the choice made for $A_1$ and $A_2$) via the Hamiltonian formulation of classical mechanics, where $q$ and $r$ are the conjugate variables.

As is the case for all nonlinear partial differential equations solvable by the CIST, (1) appears as the compatibility condition for two underlying linear equations,

$$\frac{\partial}{\partial x}\psi=J\frac{\partial}{\partial y}\psi+Q\psi, \tag{3a}$$

$$\frac{\partial}{\partial t}\psi=A\psi+iQ\frac{\partial}{\partial y}\psi+iJ\frac{\partial^2}{\partial y^2}\psi, \tag{3b}$$

where

$$Q=\begin{pmatrix}0 & q\\ r & 0\end{pmatrix},\quad J=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{4}$$

$$A=\begin{pmatrix}A_1 & \frac{1}{2}i(q_x-q_y)\\ -\frac{1}{2}i(r_x-r_y) & A_2\end{pmatrix}, \tag{5}$$

and $\psi=\psi(x,y,t)$ is a $2\times 2$ solution matrix.

The first of these equations, (3a), can be viewed simply as a linear scattering problem in which $q$ plays the role of the potential. (3a), for suitable choice of boundary conditions, can be rewritten as a system of linear integral solutions,

$$\tilde{\psi}_{ij}(\xi,\kappa,\lambda)=\delta_{ij}\exp[2i(\kappa_R+\lambda)J_j\xi_j]+\int\int d\xi'\,G^L_{ij}(\xi-\xi',\kappa)[Q(\xi')\tilde{\psi}(\xi',\kappa,\lambda)]_{ij}, \tag{6}$$

where $\xi_1=x+y$ and $\xi_2=x-y$ with $\xi$ denoting the coordinate pair $(\xi_1,\xi_2)$, $\kappa=\kappa_R+i\kappa_I$ is a complex parameter, $\lambda$ is a real parameter, the indices $i,j$ can each take on values 1 or 2 (where we use the notation $\bar{1}\equiv 2$ and $\bar{2}\equiv 1$), and all integrations are over infinite space. Also, for convenience of notation we use

$$\tilde{\psi}_{ij}(\xi,\kappa,\lambda,t)=\psi_{ij}(\xi,\kappa,\lambda,t)\exp[i(\kappa_R+\lambda)^2J_jt],$$

and we shall suppress the argument, $t$.

We choose the Green's function

$$G^L_{ij}(\xi,\kappa)=G_i(\xi,\hat{\kappa}_{ij})=\int\frac{dl}{2\pi}\exp[2i(\hat{\kappa}^R_{ij}+l)J_i\xi_i][\theta(\xi_1+\xi_2)\theta(-J_il)-\theta(-\xi_1-\xi_2)\theta(J_il)], \tag{7}$$

with $\hat{\kappa}_{ij}=\hat{\kappa}^R_{ij}+i\hat{\kappa}^I_{ij}$, $\hat{\kappa}^R_{ij}=\kappa_I+J_iJ_j(\kappa_R-\kappa_I)_1$, and $\hat{\kappa}^I_{ij}=\kappa_I$. $G^L(\xi,\kappa)$ is obtained by taking the appropriate limit of the Green's function of the more general $D$-bar problem.[10]

We also will find it useful to define a solution, $\zeta$, of an adjoint linear problem,

$$\tilde{\zeta}_{ik}(\xi,\hat{\kappa}_{kj},\lambda')=\delta_{ik}\exp[-2i(\hat{\kappa}^R_{kj}+\lambda')J_k\xi_k]+\int\int d\xi'\sum_{l=1}^{2}\tilde{\zeta}_{il}(\xi',\hat{\kappa}_{lj},\lambda')Q_{lk}(\xi')G_k(\xi'-\xi,\hat{\kappa}_{kj}). \tag{8}$$

Of fundamental interest in both the classical and the quantum problem is the "scattering matrix" or the "scattering data" of (6), which we define to be

$$T_{ij}(\kappa,\lambda,\lambda')=\int\int d\xi\exp[-2i(\hat{\kappa}^R_{ij}+\lambda')J_i\xi_i][Q(\xi)\tilde{\psi}(\xi,\kappa,\lambda)]_{ij}. \tag{9}$$

For certain choices of the parameters $\kappa$, $\lambda$, and $\lambda'$, $T$ can be shown to have a very simple time dependence, and is thus used in the CIST to "reconstruct" the potential $q(x,y,t)$ at arbitrary times, for appropriately given initial conditions.

We can calculate Poisson-bracket relations between elements of $T$, where we define canonical Poisson brackets

$$\{f,g\} = i \int \int d\xi \left[ \frac{\delta f}{\delta q(\xi)} \frac{\delta g}{\delta r(\xi)} - \frac{\delta f}{\delta r(\xi)} \frac{\delta g}{\delta q(\xi)} \right]. \tag{10}$$

We find, by use of the linear integral equations, (6) and (8), that

$$\{T_{\alpha\beta}(\kappa,\lambda,\lambda'),T_{\gamma\delta}(\tau,\mu,\mu')\} = \sum_{a=1}^{2} \int \int d\xi \, \tilde{\zeta}_{\alpha a}(\xi,\hat{\kappa}_{\alpha\beta},\lambda')\tilde{\psi}_{\bar{a}\beta}(\xi,\kappa,\lambda)\tilde{\zeta}_{\gamma\bar{a}}(\xi,\tau_{\bar{a}\delta},\mu')\tilde{\psi}_{a\delta}(\xi,\tau,\mu). \tag{11}$$

The solution $\tilde{\psi}$ and its adjoint $\tilde{\zeta}$ satisfy

$$\sum_{k=1}^{2} \frac{\partial}{\partial \xi_k} \tilde{\zeta}_{ik}(\xi,\hat{\kappa}_{kj},\lambda')\tilde{\psi}_{kj}(\xi,\tau,\mu) = 0.$$

This identity can be used to rewrite the integrand appearing in (11) as

$$\begin{aligned} &\{T_{\alpha\beta}(\kappa,\lambda,\lambda'),T_{\gamma\delta}(\tau,\mu,\mu') \\ &\quad = -\sum_{a=1}^{2} \int d\xi_a \int d\xi_a' \, \theta(J_a(\xi_a-\xi_a'))\tilde{\zeta}_{\alpha a}(\xi_a,\xi_{\bar{a}},\hat{\kappa}_{\alpha\beta},\lambda')\tilde{\psi}_{a\beta}(\xi_a',\xi_{\bar{a}},\kappa,\lambda)\zeta_{\gamma a}(\xi_a',\xi_{\bar{a}},\hat{\tau}_{a\delta},\mu')\tilde{\psi}_{a\delta}(\xi_a,\xi_{\bar{a}},\tau,\mu) \Big|_{\xi_a^-=-\infty}^{\xi_a^-=+\infty} \\ &\qquad + \sum_{a=1}^{2} J_a \int d\xi_a \, \tilde{\zeta}_{\alpha a}(\xi,\hat{\kappa}_{\alpha\beta},\lambda')\tilde{\psi}_{a\delta}(\xi,\tau,\mu) \Big|_{J_a^-\xi_a^-=\infty} \int d\xi_{\bar{a}} \, \tilde{\zeta}_{\gamma\bar{a}}(\xi',\hat{\tau}_{\bar{a}\delta},\mu')\tilde{\psi}_{\bar{a}\beta}(\xi',\kappa,\lambda) \Big|_{J_a\xi_a'=-\infty}. \end{aligned} \tag{12}$$

In order to evaluate (12) it is necessary to find asymptotic expressions for $\tilde{\psi}$ and $\tilde{\zeta}$. However, these can be found easily by our using (6) and (8) and noting that it is possible to write $G^L$ in the two alternative forms (7b) or (7c). Then

$$\lim_{\xi_k^- \to \pm\infty} \psi_{kj}(\xi,\kappa,\lambda) = \delta_{kj} \exp[2i(\kappa_R+\lambda)J_k\xi_k] \pm \int \frac{dl}{2\pi} \theta(\mp J_k l) \exp[2i(\hat{\kappa}_{kj}^R + l)J_k\xi_k] T_{kj}(\kappa,\lambda,l), \tag{13}$$

and

$$\lim_{\xi_k^- \to \pm\infty} \tilde{\zeta}_{ik}(\xi,\hat{\kappa}_{kj},\lambda') = \delta_{ik} \exp[-2i(\hat{\kappa}_{kj}^R+\lambda')J_k\xi_k] \mp \int \frac{dl}{2\pi} T_{ik}(\hat{\kappa}_{kj},l,\lambda')\theta(\pm J_k l) \exp[-2i(\hat{\kappa}_{kj}^R+l)J_k\xi_k]. \tag{14}$$

Inserting (13) and (14) into (12), we arrive at an expression for $\{T_{\alpha\beta}(\kappa,\lambda,\lambda'),T_{\gamma\delta}(\tau,\mu,\mu')\}$ purely in terms of $T$'s.

Instead of writing down a lengthy expression, which contains terms up to quartic in $T$, we instead give results in two interesting limiting cases. First, letting $\lambda=\lambda'=\mu=\mu'=0$ and $T(\kappa,0,0)\equiv T^L(\kappa)$, we recover the scattering data of the hyperbolic limit of the $D$-bar problem, and, making use of an identity easily derived from (6), find Poisson-bracket relations

$$\{T_{11}^L(\kappa),T_{11}^L(\tau)\} = \{T_{12}^L(\kappa),T_{12}^L(\tau)\} = \{T_{21}^L(\kappa),T_{21}^L(\tau)\} = 0, \tag{15a}$$

$$\{T_{12}^L(\kappa),T_{21}^L(\tau)\} = (2\pi)^2\delta(\kappa_R+\tau_R-\kappa_I-\tau_I)\delta(\kappa_I-\tau_I), \tag{15b}$$

$$\{T_{11}^L(\kappa),T_{12}^L(\tau)\} = \left[ \frac{i}{\kappa_R-\hat{\tau}_{12}^R-i\epsilon} + 2\pi\delta(\kappa_R-\hat{\tau}_{12}^R)\theta(\kappa_I-\tau_I) \right] T_{12}^L(\tau), \tag{15c}$$

as well as a number of other similar relations.

Alternatively, we can take the limit $\kappa_I \to +\infty$, $\kappa_R \to +\infty$, $T(\kappa,\lambda,\lambda') \to T^+(\theta,\theta')$, where $\theta=\kappa_R+\lambda$, $\theta'=\kappa_R+\lambda'$ are kept finite. In this way, we recover the scattering data associated with a solution to (3a),

$$\mu_{ij}^+(\xi,\theta) \equiv \psi_{ij}(\xi,\theta)\exp[-2i\theta J_i\xi_j + i\theta^2 J_j t],$$

analytic in the upper-half $\theta$ plane, which is used in the Riemann-Hilbert approach to CIST. We find

$$\begin{aligned} \{S_{\alpha\beta}^+(\theta,\theta'),S_{\gamma\delta}^+(\phi,\phi')\} = {} & \tfrac{1}{2} S_{\gamma\beta}^+(\phi,\theta')S_{\alpha\delta}^+(\theta,\phi')(J_\beta-J_\alpha) - \delta_{\alpha\gamma}J_\alpha \int \frac{d\sigma}{2\pi i(\sigma+i\epsilon)} S_{\alpha\beta}^+(\phi+\sigma,\theta')S_{\gamma\delta}^+(\theta-\sigma,\phi') \\ & + \delta_{\beta\delta}J_\beta \int \frac{d\sigma}{2\pi i(\sigma+i\epsilon)} S_{\gamma\delta}^+(\phi,\theta'-\sigma)S_{\alpha\beta}^+(\theta,\phi'+\sigma), \end{aligned} \tag{16}$$

where we have defined

$$S^+(\theta,\theta') = 2\pi J\delta(\theta-\theta') + T^+(\theta,\theta').$$

Similarly, the limit $\kappa_I \to -\infty$, $\kappa_R \to +\infty$ gives us the scattering data associated with a solution analytic in the lower-half plane.

From the Poisson-bracket relations for the elements of $T^L$, given by (15), one can see that $T^L_{12}(\kappa)$ and $T^L_{21}(\hat{\kappa}_{12})$ are related to the original conjugate variables, $q$ and $r$, by a canonical transformation. Furthermore, $T^L_{11}(\kappa)$ and $T^L_{22}(\kappa)$ are action variables of the theory, and it can be shown by use of the appropriate limit of the $D$-bar formalism that they can both be expanded in powers of $1/\kappa_R$, generating two infinite sets of constants of the motion, including as members the total momentum and energy Hamiltonian of the system. Thus the nonlinear evolution equation, (1) and (2), is exactly integrable, with proper boundary conditions ($D$ bar).

The results, (16), for the Riemann-Hilbert formulation of the inverse problem have a very different form and reduce in the (1+1)D limit to well-known classical relations of the YB type. This leads us to search for analogous quantum commutation relations. We have been able, in fact, to calculate such relations for an operator version of DS I with ordering taken to be as it appears in (1)-(3). (Note that we do not treat the normal-ordered case.) If we replace the canonical Poisson-bracket relations for $q$ and $r$ by their corresponding canonical equal-time commutation relations, we can modify our calculation slightly, now taking care to preserve correct operator ordering throughout, and derive commutation relations for elements of $S^+$ (or similarly for $S^-$). We find generalized Yang-Baxter commutation relations for $S^+$, given by (16), with $\{S^+, S^+\}$ replaced by $[S^+, S^+]$ on the left-hand side of the equation.

In the quantum case, as well as the classical case, it is not obvious how to identify the generators of conserved quantities in terms of $T^+$ and $T^-$. However, $T^+$ and $T^-$ are related to $T^L$ via integral equations, and the argument used in the $D$-bar approach to show that $T^L_{11}$ and $T^L_{22}$ are time independent and can be expanded in powers of $1/\kappa_R$ can also be made for the operator version of the problem.

Quantum NLS is a (1+1)D nonrelativistic theory for complex scalar bosons with a four-field interaction, or equivalently, rewritten in first-quantized form, it describes a system of bosons interacting via a two-body $\delta$-function potential. The exact energy spectrum can be calculated by use of Bethe's *Ansatz,* and this fact is intimately connected to the existence of a corresponding YB algebra.[9] In a sense, quantum NLS serves as a paradigm for other integrable quantum field theories in (1+1)D, such as the Thirring model. We believe that the same will be true for its (2+1)D generalization, Davey-Stewartson, and that the generalized YB relations will be useful in developing a generalization of Bethe's *Ansatz* to extract the physical energy spectrum of such models, and also exact $S$ matrices.

This work was partially supported by the National Science Foundation under Grants No. DMS-8507658 and No. DMS-8202117, the Office of Naval Research under Grant No. N00014-76-C-0867, and the Air Force Office of Scientific Research under Grant No. AFOSR-84-0005. One of us (C.L.S.) would like to thank H. H. Chen for his hospitality at the Laboratory for Plasma and Fusion Energy Studies, University of Maryland, where part of the work was completed.

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