Nonlinear State of $m = 1$ Instability in Tokamaks with Nonmonotonic q Profiles

Avinash,[a] R. J. Hastie, and J. B. Taylor
United Kingdom Atomic Energy Authority–Euratom Fusion Association, Culham Laboratory, Abingdon, Oxfordshire OX14 3DB, England

and

S. C. Cowley
Princeton Plasma Physics Laboratory, Princeton, New Jersey 08544
(Received 24 August 1987)

The nonlinear saturated state of the $m=1$, $n=1$ ideal MHD instability is calculated for a large-aspect-ratio tokamak. When the $q(r)$ profile is nonmonotonic with $\Delta q = q_{\min} - 1 > 0$, the amplitude ξ of the nonlinear state is given by $\xi^2 q''/\Delta q = 7.9[(\Delta q_c/\Delta q)^{3/2} - 1]$, where Δq_c is the critical value of Δq at which the system is marginally stable. This nonlinear state is similar to that seen during the sawtooth crash in large tokamaks and may be related to the steady-state oscillations seen when sawteeth are suppressed by lower-hybrid current drive.

PACS numbers: 52.55.Fa, 52.30.Bt, 52.55.Dy

Helical perturbations with mode numbers $m=1$, $n=1$ are a very common feature of toroidal plasma discharges. In particular they appear to be an essential ingredient of the ubiquitous sawtooth relaxation oscillations in tokamaks.[1,2] They are also seen in computer simulations of tokamak discharges.[3]

Although the linear theory of these perturbations has been very fully explored, there are few analytic calculations of their nonlinear form. In this Letter we report the calculation of the nonlinear state of the $m=1$, $n=1$ ideal-fluid instability in a toroidal discharge and comment on its possible relevance to some observations on tokamak sawteeth.

The nonlinear state of an $m=1$, $n=1$ ideal-fluid instability in a tokamak with a monotonically increasing $q(r)$ profile and a resonant surface at $q=1$ was calculated by Rosenbluth, Dagazian, and Rutherford[4] some time ago. (Here q is the toroidal winding number, or "safety factor," of the magnetic field line; in a cylindrical approximation $q = rB_z/RB_\theta$.) However, for this resonant q profile the ideal-fluid calculation leads to a singular current layer on the $q=1$ surface—which indicates that nonideal resistive effects must be important. These may lead to a completely different state.[5]

Here we consider a discharge with a nonmonotonic $q(r)$ profile, with $q_{\min}$ above, but close to, unity (Fig. 1). In this case the nonlinear state involves no singular currents. The ideal-fluid calculation is self-contained and can be calculated exactly in an asymptotic expansion. We find that the amplitude of the saturated helical deformation ξ, inside the $q_{\min}$ surface, is given by

$$\xi^2 q''/\Delta q = 7.9[(\Delta q_c/\Delta q)^{3/2} - 1], \tag{1}$$

where $\Delta q = q_{\min} - 1$ and Δq_c is the critical value of Δq at which the discharge is marginally stable. This helical state agrees well with the computer simulations[3] where just such a nonmonotonic q profile was used to simulate the helical deformations seen in the Joint European Torus (JET).

Before discussing the nonlinear calculation, we recall some properties of the linear $m=1$, $n=1$ instability. The stability of a large-aspect-ratio *toroidal* plasma was first correctly analyzed by Bussac *et al.*[6] Their analysis was recently extended by Hastie *et al.*[7] who showed that a profile such as Fig. 1 is unstable to $m=1$ perturbations when Δq is less than the critical value

$$\Delta q_c = (2r_1^2 q'')^{1/3} (\pi r_1^2 |\delta W^T| / R^2)^{2/3}, \tag{2}$$

where q'' is evaluated at r_1 and R is the major radius. The (negative) energy δW^T is given in Refs. 6 and 7. (This energy is the basic parameter of linear theory and can be regarded as a known input for the nonlinear calculation.) When Δq is small, the growth rate of the instability is given by

$$\bar{\gamma}\left(1 + \frac{\Delta q}{\bar{\gamma}}\right)^{1/3} = \frac{1}{(r_1^2 q'')^{1/3}} \left(\frac{\pi r_1^2 |\delta W^T|}{R^2}\right)^{2/3}, \tag{3}$$

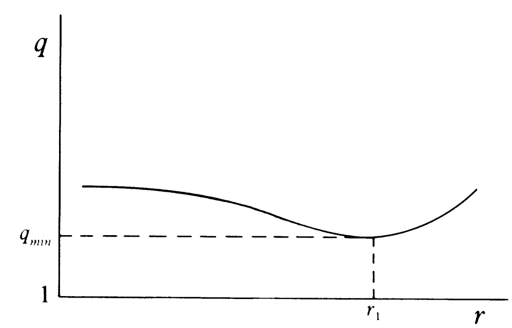

FIG. 1. Nonmonotonic $q(r)$ profile.

where $\bar{\gamma}^2 = 3\gamma^2/\omega_A^2 + \Delta q^2$ and ω_A is the toroidal Alfvén frequency.

In the linear calculation the parameters $\epsilon \equiv r_1/R$, γ/ω_A, and Δq are small and an optimal ordering $\gamma/\omega_A \simeq \Delta q \simeq \epsilon^{4/3}$ is used. The lowest-order perturbation is a rigid displacement for $r < r_1$ and there is a narrow "inner" layer around r_1, of width $\Delta r_1 \simeq \epsilon^{2/3} r_1$, in which plasma inertia is important. The perturbation in this inertial layer is matched to that in the outer regions where inertia is negligible. In the matching region the $m=1$ component of the outer perturbation is[8] $\xi_{ext} = \xi_0 + \xi_2$ with $\xi_0 = \text{const}$ for $r < r_1$, $\xi_0 = 0$ for $r > r_1$, and

$$\frac{r_1}{\xi_0}\frac{d\xi_2}{dr} = \frac{r_1^2}{R^2}\frac{\delta W^T}{(q-1)^2}. \tag{4}$$

We now consider the nonlinear saturated state of this instability. Following Ref. 4, we search for a finite-amplitude equilibrium which is accessible from the linearly unstable one. The problem can again be separated into an inner layer around r_1 and two outer regions. However, the inner layer is now determined by nonlinear effects rather than inertia. With the ordering used in the linear analysis, nonlinear effects can be neglected in the outer regions so long as the amplitude of the deformation ξ is of order ϵ or smaller. (We shall return to this point later.) The perturbation in the outer regions is then of the same form as in linear theory.

The nonlinear effects in the inner layer are important for all (near) resonant harmonics with $m/n = 1$, so that we must consider a nonlinear helical equilibrium in the layer. This is governed by

$$\nabla^2 \psi = J_z(\psi), \tag{5}$$

where ψ is the helical flux function $(krA_\theta - A_z)$. Because of the rapid radial variation in the layer, Eq. (5) can be integrated to give[4]

$$(\partial\psi/\partial r)^2 = F(\psi) + G(\theta). \tag{6}$$

As in Ref. 4, we transform from ψ to the radius of the equivalent surface in the unperturbed cylindrical configuration:

$$d\psi/dx = (r_1 B_0/R)[\Delta q + \tfrac{1}{2}x^2 q''], \tag{7}$$

with $r = x + \xi$. Then Eq. (6) may be integrated to give

$$\xi(x,\theta) = \int_0^x \left[\frac{a^2+x^2}{(f+g)^{1/2}} - 1\right] dx + h(\theta), \tag{8}$$

where $f = f(x)$, $g = g(\theta)$, and we have introduced the parameter $a^2 = 2\Delta q/q''$.

So far Eq. (8) merely describes a helical equilibrium. To ensure that it is *accessible* from the initial unstable equilibrium we must introduce the ideal MHD constraint (flux conservation), $\oint r\,dr\,d\theta = \text{const}$. This leads to

$$\frac{1}{2\pi}\oint \frac{d\theta}{(f+g)^{1/2}} = \frac{1}{(a^2+x^2)}, \tag{9}$$

and for $|x| \to \infty$, $f(x) \to x^4$. Hence the nonlinear displacement has the form

$$\xi(x,\theta) \to h(\theta) + \frac{g(\theta)}{6x^3} \pm \int_0^\infty dx \left[\frac{a^2+x^2}{(f+g)^{1/2}} - 1\right], \tag{10}$$

for $x \to \pm\infty$. This must now be matched to the solution in the outer regions [Eq. (4)].

We consider first matching the dominant $m=1$ Fourier component. This requires that

$$h(\theta) = -\int_0^\infty dx \left[\frac{a^2+x^2}{(f+g)^{1/2}} - 1\right] = \frac{\xi_0}{2}\cos\theta, \tag{11}$$

and

$$\frac{1}{2\pi}\oint g(\theta)\cos\theta = -\frac{4\xi_0}{r_1}\left(\frac{r_1}{R}\right)^2 \frac{\delta W^T}{(q'')^2}. \tag{12}$$

The calculation of the accessible nonlinear equilibrium thus becomes one of finding functions $f(x)$ and $g(\theta)$ satisfying Eqs. (9) and (11). Then (12) determines the amplitude of the nonlinear state. It is convenient to replace f by $a^4\tilde{f}$, g by $a^4\tilde{g}$, and x by $a\tilde{x}$; then, in nondimensional form, these equations become

$$\int_0^\infty d\tilde{x}\left[\frac{1+\tilde{x}^2}{(\tilde{f}+\tilde{g})^{1/2}} - 1\right] = -\mu\cos\theta, \tag{13a}$$

$$\frac{1}{2\pi}\oint d\theta \frac{1}{(\tilde{f}+\tilde{g})^{1/2}} = \frac{1}{1+\tilde{x}^2}, \tag{13b}$$

$$\frac{1}{2\pi}\oint d\theta\, \tilde{g}(\theta)\cos\theta = -\lambda, \tag{13c}$$

where $\mu^2 \equiv \xi^2 q''/(8\Delta q)$ and $\lambda = \xi_0 r_1 \delta W^T/(R\Delta q)^2$. We take $\oint \tilde{g}\,d\theta = 0$.

For the case of the resonant $q(r)$ profile discussed in Ref. 4, the equations corresponding to (13) could be solved only by taking a trial function for $\tilde{g}(\theta)$. It is therefore surprising that the present problem can be solved by an asymptotic expansion consistent with the ordering already adopted. In this ordering, μ is a small quantity $\simeq \epsilon^{1/3}$, and after considerable algebra one finds, up to $O(\mu^3)$,

$$\tilde{g}(\theta) = (8\mu/\pi)\cos\theta + 15(\mu/\pi)^2\cos 2\theta + \tfrac{1}{8}(\mu/\pi)^3[639\cos\theta + 135\cos 3\theta], \tag{14}$$

and

$$\tilde{f}(\tilde{x}) = (1+\tilde{x}^2)^2 + (\mu/\pi)^2[24/(1+\tilde{x}^2)^2]. \tag{15}$$

Finally, the amplitude of the nonlinear displacement is [with use of Eq. (2)]

$$\frac{\xi^2 q''}{\Delta q} = \frac{8}{71}\left(\frac{8\pi}{3}\right)^2\left[\left(\frac{\Delta q_c}{\Delta q}\right)^{3/2} - 1\right]. \qquad (16)$$

This expresses the nonlinear saturated state in terms of the strength of the instability $\Delta q_c/\Delta q$.

There are some important points to be made about the validity of Eq. (16). In deriving it, we have neglected all nonlinear contributions from the outer regions and matched only the $m=1$ Fourier components of the linear displacement. With the ordering $\epsilon \equiv r_1/R$, $\gamma/\omega_A \simeq \Delta q \simeq \epsilon^{4/3}$ and an amplitude $\xi(q'')^{1/2} \simeq \epsilon$, the nonlinear effects in the outer region are *formally* larger than those in the inner layer. However, they vanish identically because the lowest-order displacement ξ_0 is essentially a rigid shift leaving the plasma in equilibrium. Consequently, when the $m \neq 1$ Fourier components are included in the matching one finds only small corrections to Eq. (16). (These aspects of the problem will be discussed elsewhere.) In fact, Eq. (16) is asymptotically correct with the stated ordering. We now see that for consistency this requires $\Delta q_c - \Delta q$ to be of higher order than Δq_c itself; i.e., the original equilibrium must be close to marginal stability. Then the nonlinear amplitude can also be expressed in terms of the linear instability growth rate as

$$\xi^2 q'' = \frac{15}{71}\left(\frac{8\pi}{3}\right)^2 \frac{\gamma^2}{\omega_A^2}. \qquad (17)$$

In summary, we have calculated the nonlinear state of the unstable $m=1$, $n=1$ ideal MHD instability in a large-aspect-ratio tokamak with a nonmonotonic $q(r)$ profile. This nonlinear state involves no singular current layer and can therefore be considered complete within ideal theory. Of course, although the general behavior of the perturbed energy indicates[4] that this nonlinear state should be stable against further $m=1$, $n=1$ deformation, the question of its stability against other forms of perturbation is an important question requiring further investigation.[9]

The amplitude of this nonlinear state agrees well with the plasma simulations of JET.[3] It may also account for certain observations of precursor oscillations during the sawtooth ramp on other tokamaks. In particular, the observations on PETULA [10] and PLT,[11] when lower-hybrid current drive was employed to stabilize the sawtooth crashes, can be interpreted in this way. At intermediate power levels, just sufficient to prevent the sawtooth crashes, the precursor oscillations of the ramp phase develop into steady state $m=1$, $n=1$ oscillations. (Similar steady $m=1$, $n=1$ oscillations have been observed[12] on JET during off-axis ion-cyclotron-resonance heating and in DOUBLET III*A* under conditions of axial impurity concentration.[13]) At still higher rf power levels on PETULA,[10] these oscillations disappeared altogether. Hitherto these observations have been interpreted as stabilization of a resistive mode "island" in a profile with $q=1$ at the island radius. However, they might equally be interpreted as the gradual reduction in the amplitude of a nonlinear ideal MHD instability in a profile such as Fig. 1 as Δq increases towards Δq_c.

One of us (Avinash) wishes to acknowledge the hospitality of the Culham Laboratory during this work.

[(a)]Permanent address: Institute for Plasma Research, Gandhinagar Highway Bhat, Gandhinagar 382 424, India.

[1]S. von Goeler, W. Stodiek, and N. Sauthoff, Phys. Rev. Lett. **33**, 1201 (1974).

[2]A. W. Edwards *et al.*, Phys. Rev. Lett. **57**, 210 (1986).

[3]L. A. Charlton *et al.*, to be published.

[4]M. N. Rosenbluth, R. Y. Dagazian, and P. H. Rutherford, Phys. Fluids **16**, 1894 (1973).

[5]R. D. Hazeltine, J. D. Meiss, and P. J. Morrison, Phys. Fluids **29**, 1633 (1986).

[6]M. N. Bussac, R. Pellat, D. Edery, and J. L. Soule, Phys. Rev. Lett. **35**, 1638 (1975).

[7]R. J. Hastie *et al.*, Phys. Fluids **30**, 1756 (1987).

[8]G. Ara *et al.*, Ann. Phys. (N.Y.) **112**, 443 (1978).

[9]M. N. Bussac and R. Pellat, following Letter [Phys. Rev. Lett. **59**, 2650 (1987)].

[10]D. van Houtte *et al.*, in *Proceedings of the Thirteenth European Conference on Controlled Fusion and Plasma Heating, Schliersee, West Germany, 1986,* edited by G. Briffod and M. Kaufmann (European Physical Society, Petit-Lancy, Switzerland, 1986), Vol. 2, p. 331.

[11]T. K. Chu *et al.*, Nucl. Fusion **26**, 666 (1986).

[12]D. C. Robinson, P. Haynes, and A. W. Morris, in *Magnetic Reconnection and Turbulence,* edited by M. A. Dubois, D. Grésillon, and M. N. Bussac (Les Editions de Physique, Les Ulis, France, 1986), p. 259.

[13]G. L. Jahns *et al.*, Nucl. Fusion **22**, 1049 (1982).