Decay Width and Shift of a Quasistationary State

S. A. Gurvitz
Weizmann Institute of Science, 76 100 Rehovot, Israel

and

G. Kalbermann
The Hebrew University of Jerusalem, 91 904 Jerusalem, Israel
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We develop a two-potential approach to the decay of a quasistationary state. The method enables us to obtain a simple algebraic formula for the decay width and the energy shift of a metastable state. The quasiclassical limit of the width found leads to the well-known Gamow formula, this time with a well-defined preexponential factor. The energy shift together with its quasiclassical limit is obtained in closed form.

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One of the textbook problems in quantum mechanics[1] is that of the lifetime of metastable states. Metastable states arise as resonances in scattering reactions in a process that starts with a free wave at infinity impinging on a potential that distorts the wave and eventually traps it in a quasibound state for a sizable amount of time, and finally the decay occurs. It can be shown[1] that the resonance generation process does not affect the decay properties for long-lived states. Because of this fact and the complications arising from the treatment of the full reaction, it is preferable to deal directly with the decay to the continuum of the quasibound state regarded at some initial time as a true bound state. The latter coincides with nuclear radioactivity[2] and is of great interest on its own. In the following we focus on the problem of the decay of a prepared state.

The decay width of a metastable state in the quasiclassical limit was found long ago by Gamow.[3] Even in this treatment the preexponential factor appearing in the width formula was hard to estimate except for the case of high-lying states. Since then, little progress has been made towards finding a general formula that embodies the quasiclassical result of Gamow including the energy shift in closed form.

We present here a simple algebraic expression for the decay width and the energy shift of a metastable state in a potential barrier of any shape, a typical example of which is depicted in Fig. 1(a). The quasistationary state possesses an energy $E < V(R) = V_0$ very close to E_0, that of the bound state generated by $U(r)$ of Fig. 1(b) [$U(r) = V(r)$ for $r \leq R$ and $U(r) = V(R) = V_0$ for $r > R$]; therefore we split $V(r)$ into $U(r) + W(r)$ [Fig. 1(c)], where $W(r) = 0$ for $r \leq R$, and $W(r) = V(r) - V_0$ for $r > R$. A two-potential formalism emerges straightforwardly.

Consider the state $|\Phi_0\rangle$ which is an eigenstate of $H_0 = -\nabla^2/2m + U(r)$ with eigenvalue E_0 [Fig. 1(b)]. At $t=0$ we switch on the distorting potential $W(r)$ of Fig. 1(c). $|\Phi_0\rangle$ is no longer an eigenstate of the full Hamiltonian $H = H_0 + W(r)$. The wave function is now

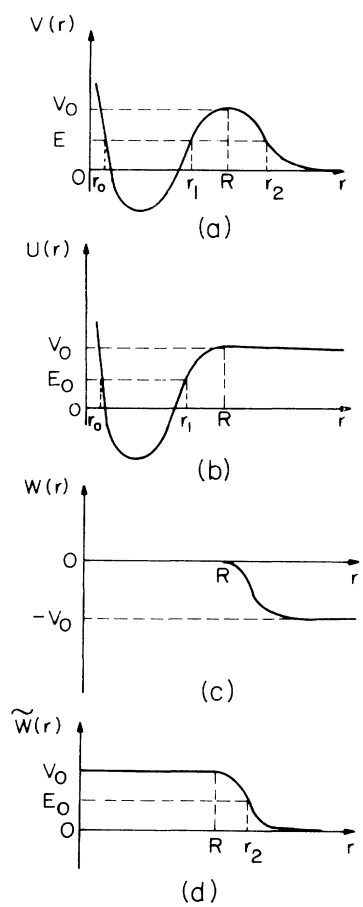

FIG. 1. The potentials V, U, W, and $\tilde{W}$. r_0, r_1, r_2 are the quasiclassical turning points.

expanded as

$$\Phi_0(r,t)=b_0(t)e^{-iE_0t}\Phi_0(r)+\int d^3k(2\pi)^{-3}b_{\mathbf{k}}(t)e^{-iE_kt}\Phi_{\mathbf{k}}(r) \tag{1}$$

in terms of bound (continuum) wave functions Φ_0 ($\Phi_{\mathbf{k}}$) which are eigenstates of H_0. Equation (1) is supplemented with the condition $b_0(0)=1$, $b_{\mathbf{k}}(0)=0$. For the sake of simplicity we considered a case where the spectrum of H_0 contains only one bound state. Inserting Eq. (1) in the Schrödinger equation we obtain

$$i\frac{db_0(t)}{dt}=b_0(t)\langle\Phi_0|W|\Phi_0\rangle+\int d^3k(2\pi)^{-3}b_{\mathbf{k}}(t)e^{i(E_0-E_k)t}\langle\Phi_0|W|\Phi_{\mathbf{k}}\rangle, \tag{2a}$$

$$di\frac{db_{\mathbf{k}}(t)}{dt}=b_0(t)\langle\Phi_{\mathbf{k}}|W|\Phi_0\rangle e^{i(E_k-E_0)t}+\int d^3k'(2\pi)^{-3}b_{\mathbf{k}'}(t)e^{i(E_k-E_{k'})t}\langle\Phi_{\mathbf{k}}|W|\Phi_{\mathbf{k}'}\rangle. \tag{2b}$$

If $b_0(t)$ as a solution of Eqs. (2) is found such that for large times it drops as $e^{-\Gamma t}$ we are confronted with a metastable or resonant state.

An obvious problem which arises in Eq. (2b) is the noncompactness of W, since $W(r)\to -V_0$ for $r\to\infty$ [Fig. 1(c)]. Therefore singular pieces are generated in the matrix elements describing continuum-to-continuum transitions. In order to solve this problem we introduce the potential $\tilde{W}(r)=W(r)+V_0$ that vanishes for $r\to\infty$ [Fig. 1(d)]. We now substitute

$$\langle\phi_{\mathbf{k}}|W|\phi_{\mathbf{k}'}\rangle=\langle\phi_{\mathbf{k}}|\tilde{W}|\phi_{\mathbf{k}'}\rangle-(2\pi)^3V_0\delta(\mathbf{k}-\mathbf{k}')$$

into Eq. (2b), and also define $b_{\mathbf{k}}(t)=e^{iV_0t}\tilde{b}_{\mathbf{k}}(t)$. Equations (2) become

$$i\frac{db_0(t)}{dt}=b_0(t)\langle\Phi_0|W|\Phi_0\rangle+\int d^3k(2\pi)^{-3}\tilde{b}_{\mathbf{k}}(t)e^{i(E_0+V_0-E_k)t}\langle\Phi_0|W|\Phi_{\mathbf{k}}\rangle, \tag{3a}$$

$$i\frac{d\tilde{b}_{\mathbf{k}}(t)}{dt}=b_0(t)\langle\Phi_{\mathbf{k}}|W|\Phi_0\rangle e^{i(E_k-V_0-E_0)t}+\int d^3k'(2\pi)^{-3}\tilde{b}_{\mathbf{k}'}(t)e^{i(E_k-E_{k'})t}\langle\Phi_{\mathbf{k}}|\tilde{W}|\Phi_{\mathbf{k}'}\rangle. \tag{3b}$$

The Fermi "golden rule" can be now obtained by neglect of the continuum-to-continuum matrix elements in Eq. (3b), namely

$$\Gamma=2\pi\int|\langle\Phi_0|W|\Phi_{\mathbf{k}}\rangle|^2\rho(E_k)\delta(E_0+V_0-E_k)dE_k, \tag{4}$$

ρ being the density of final states and $E_k=V_0+k^2/2m$. Note that the same approximation in Eq. (2b) yields an erroneous result for the Fermi "golden rule" that has the wrong energy-conservation condition.

In order to account for the higher-order terms in Eq. (3b) we solve Eqs. (3) exactly. The simplest way of obtaining the solution of such equations is the Laplace transformation $b_0(t)\to b_0(\epsilon)$.[1] After some algebra we find

$$-ib_0(\epsilon)=(\epsilon-\langle\Phi_0|W|\Phi_0\rangle-\langle\Phi_0|W\tilde{G}W|\Phi_0\rangle)^{-1}, \tag{5}$$

where $\tilde{G}$ is defined by

$$\tilde{G}=[(1-\Lambda)/(\epsilon+E_0+V_0-H_0)](1+\tilde{W}\tilde{G}). \tag{6}$$

Here $\Lambda=|\Phi_0\rangle\langle\Phi_0|$ is the projection operator onto the state $|\Phi_0\rangle$. The value of $\epsilon=\epsilon_0$ for which a pole in $b_0(\epsilon)$ arises determines the shift $\Delta=\mathrm{Re}(\epsilon_0)$ and the width $\Gamma=-2\,\mathrm{Im}(\epsilon_0)$ of the metastable state; this can be seen[1] by our taking the inverse Laplace transform of $b_0(\epsilon)$ and looking for the leading contribution to the integral as $t\to\infty$. This pole appears in the second Riemann sheet in the complex ϵ plane. Therefore ϵ_0 is defined by the equation

$$\epsilon_0=\langle\Phi_0|W|\Phi_0\rangle+\langle\Phi_0|W\tilde{G}W|\Phi_0\rangle. \tag{7}$$

One can easily recover the Fermi "golden rule" [Eq. (4)] from Eq. (7) if one takes $\tilde{W}=0$ in Eq. (6).

In fact, we can derive Eq. (7) using the procedure of Goldberger and Watson,[1] by considering the resonance as pole in a matrix element $\langle\phi_0|G|\phi_0\rangle$ of the full Green's function $G=[E+\nabla^2/2m-V(r)]$ at $E=\epsilon_0+E_0$. Then introducing the level shift operator $R=\tilde{W}+\tilde{W}\tilde{G}\tilde{W}$ and using the orthogonality of eigenstates of H_0, so that $\tilde{G}\tilde{W}|\phi_0\rangle=\tilde{G}W|\phi_0\rangle$, we obtain Eq. (7) through a relation between G, $\tilde{G}$, and R. A detailed derivation will be given elsewhere. Here we only mention that the validity of Eq. (7) does not depend on how many bound states the potential $U(r)$ has. Also we mention that the position of pole in G depends only on the potential $V(r)$. Therefore, the value ϵ_0+E_0 would not depend on a particular decomposition of the potential V into $U+W$.

Expanding $|\Phi_0\rangle$ in partial waves and including the centrifugal contribution to the potential in $U(r)$, for spherically symmetric potentials, we rewrite Eq. (7) in terms of the radial wave function φ_0, partial-wave index suppressed:

$$\epsilon_0=\int_R^\infty |\varphi_0(r)|^2 W(r)dr+\int_R^\infty dr\int_R^\infty dr'\varphi_0(r)W(r)\tilde{G}(E,r,r')W(r')\varphi_0(r'), \tag{8}$$

where $E=E_0+\epsilon_0$.

Our main approximation at this stage consists in replacing $\tilde{G}$ by $G_{\tilde{W}}=[E+\nabla^2/2m-\tilde{W}]^{-1}$, the propagator corresponding to the interaction $\tilde{W}(r)$ [Fig. 1(d)]. We proceed in this manner for the following reasons: It is only the projection operator Λ which makes a difference between $\tilde{G}$ and $G=[E+\nabla^2/2m-V]^{-1}$. The integration region in Eq. (7) starts at $r,r'\geq R$, where the contribution from Λ is not important. Therefore $\tilde{G}$ can be replaced by $G_{\tilde{W}}$, since $G_{\tilde{W}}$ and G obey the same differential equation because $\tilde{W}(r)=V(r)$ for $r\geq R$. The differences between $\tilde{G}$ and $G_{\tilde{W}}$ arising from the inner part $r<R$ are minute, since both $\tilde{G}$ and $G_{\tilde{W}}$ lack any resonances at E_0: $\tilde{G}$ because of the subtraction of Λ in Eq. (6) and $G_{\tilde{W}}$ because $\tilde{W}(r)=V_0$ for $r<R$ has no pocket as does $U(r)$ [see Figs. 1(a), 1(b)].[4] The nonresonance wave functions in the inner region $r<r_1$, Fig. 1(a), are suppressed approximately by a factor $\exp\{-\int_{r_1}^{r_2}[2m(V-E_0)]^{1/2}dr\}$ and therefore the error induced by the replacement $\tilde{G}$ by $G_{\tilde{W}}$ is of the order of Γ^2 (or ϵ_0^2).[5] The exact estimation of the correction term can be done by expansion of $\tilde{G}$ in terms of $G_{\tilde{W}}$. After some lengthy algebra, which will be presented elsewhere, we found that the correction term to our approximation, $\tilde{G}\cong G_{\tilde{W}}$, is indeed of order ϵ_0^2 in accordance with intuitive arguments given above.

Thus omitting higher-order terms in ϵ_0 we replace $E=E_0$ in the Green's function of Eq. (8). The partial-wave resolvent is

$$G_{\tilde{W}}(E_0,r,r')=-\frac{2m}{rr'k}\chi_k^{(+)}(r_>)\chi_k(r_<). \tag{9}$$

Here χ_k ($\chi_k^{(+)}$) is the regular (outgoing) eigenstate of the Hamiltonian $-\nabla^2/2m+\tilde{W}(r)$, with an asymptotic form $\sin(kr-\pi l/2+\delta_l)$ ($\exp[i(kr-\pi l/2+\delta_l)]$), where $k=(2mE_0)^{1/2}$, $r_>\equiv\max\{r,r'\}$, and $r_<\equiv\min\{r,r'\}$. Inserting Eq. (9) in Eq. (8) and using $\operatorname{Im}\chi_k^{(+)}=\chi_k$ we find

$$\Gamma=\frac{4m}{k}\left|\int_R^\infty \varphi_0(r)W(r)\chi_k(r)dr\right|^2, \tag{10}$$

a formula which resembles the well-known result for an isolated Breit-Weigner resonance.[6] Equation (10) is usually treated by means of some approximation scheme or else by numerical integration. In our case, however, we carry out an integration in Eq. (10) analytically with no approximations. First we note that $\varphi_0(r)=\varphi_0(R)\times\exp[-\alpha(r-R)]$ for $r\geq R$, where $\alpha=[2m(V_0-E_0)]^{1/2}$. Then taking advantage of the fact that χ_k is an eigenfunction of the Schrödinger equation for the potential $\tilde{W}(r)=W(r)+V_0$ we can replace $W(r)\chi_k(r)$ by $[E_0-V_0+(1/2m)d^2/dr^2]\chi_k(r)$ in Eq. (10). After integrating by parts twice, we thus obtain

$$\int_R^\infty e^{-\alpha r}W(r)\chi_k(r)dr = -\frac{e^{-\alpha R}}{2m}[\alpha\chi_k(R)+\chi_k'(R)]. \tag{11}$$

All remaining integrals cancel. Here $\chi_k'(R)$ denotes the radial derivative of χ_k at $r=R$. Substituting Eq. (11) into Eq. (10) we finally obtain a width

$$\Gamma=\frac{1}{mk}|\varphi_0(R)[\alpha\chi_k(R)+\chi_k'(R)]|^2 \cong\frac{4\alpha^2}{mk}|\varphi_0(R)\chi_k(R)|^2, \tag{12}$$

where we used $\chi_k'(R)\cong\alpha\chi_k(R)$, which is correct up to terms of order $\exp(-2\alpha R)$.

In a similar way one can find the energy shift, Δ, by performing an *exact* r,r' integration in Eq. (8). Detailed algebra will be presented elsewhere. Here we give only the final result for $\epsilon_0=\Delta-i\Gamma/2$:

$$\epsilon_0=\frac{-1}{2mk}|\varphi_0(R)|^2[\alpha\chi_k(R)+\chi_k'(R)][\alpha\chi_k^{(+)}(R)+\chi_k^{(+)\prime}(R)], \tag{13}$$

where $\chi_k^{(+)}(R)$ is value of the outgoing wave function at $r=R$. Equations (12) and (13) are the main results of this work.

We have checked Eqs. (12) and (13) for two types of potentials such as the square-well barrier[7] and the "traditional" α-decay model,[2,8] where the analytical solution of the leading terms for Γ and Δ can be found with a standard technique. Up to an accuracy of approximations used in Refs. 2, 7, and 8 the results agree *exactly*.

It is very interesting to investigate the quasiclassical limit of our result. Then the bound-state wave function is

$$\varphi_0(R)=(\sqrt{N}/2\sqrt{\alpha})\exp\left[-\int_{r_1}^R|p(r)|dr\right], \tag{14}$$

where $p(r)=\{2m[E_0-V(r)]\}^{1/2}$ and N is the quasiclassical bound-state normalization factor:

$$N\int_{r_0}^{r_1}\frac{1}{p(r)}\cos^2\left[\int_{r_0}^r p(r')dr'-\frac{\pi}{4}\right]dr=1. \tag{15}$$

The wave function $\chi_k(R)$ in the quasiclassical limit is

$$\chi_k(R)=\frac{\sqrt{k}}{2\sqrt{\alpha}}\exp\left[-\int_R^{r_2}|p(r)|dr\right]. \quad (16)$$

Substituting Eqs. (14) and (16) into Eq. (12) we obtain

$$\Gamma=(N/4m)\exp\left[-2\int_{r_1}^{r_2}|p(r)|dr\right]. \quad (17)$$

For the high-lying states where $\varphi_0(r)$ strongly oscillates one can replace the mean value of the squared cosine term in Eq. (15) by $\frac{1}{2}$. Then $N=4m/T$ where T is the quasiclassical period of motion, and we obtain the famous Gamow formula.[3] However, our result, Eq. (17), can also be used for low-lying states.

Equation (12) and its quasiclassical limit, Eq. (17), are much simpler and more general than any other results for Γ obtained in a framework of different WKB-type approximations[9] or by a path-integral technique.[10,11] We also can demonstrate that for a low-lying state, in the limit of $V(r)\gg E_0$, Eq. (17) gives the same result for Γ as that in the instanton method,[11] provided the potential $V(r)$ can be approximated by a harmonic oscillator near the minimum.

Taking the quasiclassical limit of Eq. (13) we obtain for the energy shift

$$\Delta=[NV'(R)/16\alpha^3]\exp\left[-2\int_{r_1}^{R}|p(r)|dr\right], \quad (18)$$

where $V'(R)=dV(r)/dr\,|_{r=R}$. It is interesting to note that choosing R in such a way that $V'(R)=0$, as shown in Fig. 1, we obtain that the energy shift Δ in the quasiclassical limit is zero.

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