Torsion Constraints and Super Riemann Surfaces

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Super Riemann surfaces are important in superstring theories as the generalization of the bosonic world sheet. In one approach to their study, one introduces two-dimensional supergravity, subject to certain conditions on the field strengths. Another approach builds super Riemann surfaces from superconformal patching data with no mention of the constraints. We show the equivalence of these two approaches and in particular interpret the torsion constraints as integrability conditions of a certain geometrical structure.

Bosonic string theory can be formulated in terms of quantum fields on a surface of two dimensions—a world sheet. In order to define an action for the fields, this surface must be equipped with more than just its smooth manifold structure. Namely, in the Euclidean formalism the action depends on a conformal structure; given this, we can define a Cauchy-Riemann operator $\bar{\partial}$ on the surface X and take for the classical action $\int_X \partial x^\mu \bar{\partial}^* x^\mu$, where x^μ are fields on X. In fact, every conformal structure locally looks like the usual one on the plane, but globally this need not be the case.[1]

Clearly the above remarks depend on the fact that a two-manifold with conformal structure can be regarded as having just *one* complex dimension, so that ∂x^μ can be viewed as half a volume form on X and so $\partial x^\mu \bar{\partial}^* x^\mu = (\partial_u x)(\partial_{u^*} x) du \wedge du^*$ can be integrated without specifying any additional information on X. In two or more complex dimensions no corresponding differential operator can be defined; the complex structure alone does not suffice to build a half-volume form invariantly from the first derivatives of x^μ. If we add additional data to X, for instance, a metric, then in general X is no longer locally equivalent to the standard flat space.

In this light, the corresponding supersymmetric situation may at first seem problematic. The superstring has a world sheet $\hat{X}$ with one even *and* one odd complex coordinate: u, θ. Indeed, to define an action it is not enough to supply $\hat{X}$ with a complex structure: One must give $\hat{X}$ a "superconformal structure." In this Letter we will compare two approaches to the study of such structures.

A complex manifold[2] of dimension $1|1$ with a superconformal structure is called a super Riemann surface, or SRS. Friedan defines SRS using patching data.[3] Coordinate patches of the complex plane $\mathbb{C}^{1|1}$ are glued together with use of transition functions of a very special type. Let u, θ be coordinates for $\mathbb{C}^{1|1}$. Then a coordinate transformation is permitted only if it is holomorphic, so that it transforms the vector fields $\partial/\partial u, \partial/\partial \theta$ into a linear combination of $\partial/\partial u', \partial/\partial \theta'$ (i.e., no $\partial/\partial u^*, \partial/\partial \theta^{*'}$ are allowed). Furthermore, it must specifically transform the odd vector field $D \equiv \partial/\partial \theta + \theta \partial/\partial u$ into a nonzero multiple of $D' = \partial/\partial \theta' + \theta' \partial/\partial u'$. A consistent set of such patching functions defines a SRS. (Alternatively, one may study SRS via Fuchsian groups on the super half plane.[4]) Thus it is possible that not every complex $1|1$ manifold can be made into a SRS; on the other hand, two distinct SRS could in principle be identical as complex manifolds, related by an analytic coordinate transformation which does not preserve D. In fact, only the first possibility is realized; thus the space of compact SRS sits inside the space of compact complex

manifolds.[5]

An older approach involves two-dimensional supergravity. It was originally formulated locally[6-8] and later used globally to build supermoduli space.[9] One first notes that on an ordinary Riemann surface one can specify a conformal structure by giving a metric γ_{nm}, a two-dimensional "graviton." The ∂^* operator above is then essentially the Levi-Civita covariant derivative ∇ of two-dimensional gravity.[1] ∇ is fixed uniquely by the conditions that it preserve γ and have no torsion. Using γ and ∇ we build an action reproducing the original; of course, it is invariant under those changes of γ which do not affect its conformal class, the Weyl transformations.

Thus we can think of the bosonic string world sheet as a complex manifold built from pieces of $\mathbb{C}$ by patching functions, or as a "bare" real surface with an additional globally defined structure. Each approach has its advantages.

When we try to follow the second approach into superspace, however, things are not so clear. We begin with a bare real $2\,|\,2$-dimensional manifold, for example $\mathbb{R}^{2\,|\,2}$ or more generally the surface $\hat{X}$ obtained from an ordinary surface X by choosing a spin structure.[10] X has real coordinates y^M. One then introduces a family of frames $\{E_A\}$ spanning $T\hat{X}$, where the frame index A runs over two ordinary and two odd directions: E_a are even vectors, while E_α are odd vectors. $E_A \equiv E_A{}^M \partial/\partial y^M$ thus contains sixteen real superfield degrees of freedom.

Some of the fields in E_A are gauge artifacts, just as the *vierbein* of general relativity contains spurious information as a result of the local frame invariance. Even taking this into account, however, E_A contains many more fields than the minimal graviton plus gravitino of supergravity. One is therefore led to impose *ad hoc* certain constraints on the E_A's to be considered. This program is described in four dimensions by Wess and Bagger.[11] One first introduces still more degrees of freedom in the form of a connection on $\hat{X}$, ϕ_A. One then defines the torsion and curvature of ϕ_A and imposes conditions on these tensors. These conditions in turn beget others via the Bianchi identities; remarkably in the end $E_A{}^M$ and ϕ_A are so constrained as to include only degrees of freedom corresponding to the graviton and gravitino.

In two dimensions, one has $\nabla_A = E_A + \phi_A M$, where ∂_A denotes E_A regarded as a differential operator and M is the generator of SO(2), acting on frame indices. In a complex basis M is diagonal with $M_z{}^z = -M_{z*}{}^{z*} = i$, $M_+{}^+ = -M_-{}^- = i/2$. Define the tensors T and R by $[\nabla_A, \nabla_B] = -T_{AB}{}^C \nabla_C + R_{AB} M$. One constrains the field strengths by requiring[6,7]

$$T_{ab}{}^c = T_{\alpha\beta}{}^\gamma = T_{+-}{}^z = T_{--}{}^z = 0, \quad T_{++}{}^z = -2. \quad (1)$$

(Our index notation follows Nelson and Moore.[12]) A real tensor, like T, has the property that T_{--} is the complex conjugate of $T_{++}{}^z$, and so on. The normaliza-

tion chosen here is convenient in that the frame,

$$\hat{E}_z = \frac{\partial}{\partial u}, \quad \hat{E}_+ = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial u}, \quad \phi_A \equiv 0, \quad (2)$$

satisfies (1).

The constraints (1) are a bit mysterious. We will show how they follow naturally from the imposition of a superconformal structure on $\hat{X}$. Some of the constraints will be seen to come from the requirement that $\hat{X}$ be a complex manifold, as has already been noted in D'Hoker and Phong.[13] But as we have indicated as SRS has more structure than a complex manifold, so it is not surprising that not all of the conditions arise in this way.

Let us first recast Friedan's definition slightly.[14] While the vector field $D = \partial/\partial\theta + \theta\,\partial/\partial u$ is not in general carried to itself across patch boundaries, still the space $\mathcal{D}$ spanned by D is everywhere well defined. Thus $\hat{X}$ is a complex $1\,|\,1$-dimensional manifold endowed with a holomorphic subbundle of its holomorphic tangent: $\mathcal{D} \subset T^{1,0}\hat{X}$. Such a subbundle is called a distribution. Since the Lie bracket satisfies

$$[D,D] = 2\,\partial/\partial u, \quad (3)$$

we have that $[D,D]$ together with D itself everywhere span $T^{1,0}\hat{X}$. Conversely, given a complex $\hat{X}$ with a distribution $\mathcal{D}$ satisfying this nondegeneracy condition, we can easily recover a SRS in the sense of Friedan.[15]

If we regard an ordinary Riemann surface X as a bare real manifold, then the imposition of a complex structure can be thought of as reducing its structure group from $GL(2,\mathbb{R})$ to the subgroup $GL(1,\mathbb{C})$.[16] Similarly a complex structure on $\hat{X}$ reduces $GL(2\,|\,2,\mathbb{R})$ to the subgroup $GL(1\,|\,1,\mathbb{C})$. Such a reduction can be specified by giving a frame E_A for $\hat{X}$ with the understanding that E_A is to be identified with $E_A' = U_A{}^B E_B$, where U is a function from $\hat{X}$ to $GL(1\,|\,1,\mathbb{C})$. In a complex basis such U have the block-diagonal form

$$\begin{pmatrix} E_z' \\ E_+' \\ E_{z*}' \\ E_-' \end{pmatrix} = U \begin{pmatrix} E_z \\ E_+ \\ E_{z*} \\ E_- \end{pmatrix}, \quad U = \begin{pmatrix} A & \Gamma & 0 & 0 \\ \Delta & B & 0 & 0 \\ 0 & 0 & A^* & \Gamma^* \\ 0 & 0 & \Delta^* & B^* \end{pmatrix}, \quad (4)$$

where A,B are even complex functions and Γ,Δ are odd. Clearly the action of U leaves unchanged the subspace $T^{1,0}\hat{X}$ of the complex tangent space if we take $T^{1,0}\hat{X}$ to be spanned by E_z and E_+.

Any frame thus gives rise to an "almost-complex" structure on $\hat{X}$. Unlike the case of one complex dimension, however, not every such structure actually comes from a complex manifold. For example, every complex manifold admits local analytic coordinates such that we can take $E_z = \partial/\partial u$, $E_+ = \partial/\partial\theta$; in this basis the Lie brackets all vanish: $[E_z, E_z] = [E_z, E_+] = [E_+, E_+] = 0$. Moreover, in any other basis related to this one by $GL(1\,|\,1,\mathbb{C})$ transformation we clearly have that the Lie brackets of E_z, E_+ contain no E_{z*}, E_- terms. Not every

frame has this property; thus a necessary condition for an almost-complex structure to define a complex $\hat{X}$ is

$$t_{++}{}^{z^*} = t_{++}{}^- = t_{+z}{}^- = t_{+z}{}^{z^*} = 0, \tag{5}$$

where $[E_A, E_B] = t_{AB}{}^C E_C$ defines t. [Eq. (5) is equivalent to requiring that the "Nijenhuis tensor" should vanish.] Equation (5) is satisfied by the brackets $\hat{t}_{AB}{}^C$ of the standard frame $\hat{E}_A$. Just as in the case of ordinary manifolds, (5) is, in fact, sufficient for $\hat{X}$ to be complex.[5]

In order to discuss manifolds $\hat{X}$ equipped with a distribution $\mathcal{D}$ we can again introduce frames E_A, but now only identify them under the action of a subgroup $G_1 \subset \mathrm{GL}(1 \mid 1, \mathbb{C})$. G_1 consists of matrix functions $U_A{}^B$ as in (4) but with $\Delta \equiv 0$. Then $\mathcal{D}$ is simply the space spanned by E_+; $\mathcal{D}$ is then by definition invariant under G_1 transformations. We will call a frame satisfying (5) and defined up to local G_1 transformations an "almost-superconformal structure."

When does an almost-superconformal structure come from a genuine SRS? The general theory of reductions of the structure group again gives a necessary condition for E_A to be locally equivalent to the standard $\hat{E}_A$ on a patch of $\mathbb{C}^{1|1}$.[5,16] Namely, those components of the standard $\hat{t}_{AB}{}^C$ which remain unchanged under the transformations of G_1 each give rise to a condition on the given $t_{AB}{}^C$. Since G_1 is *smaller* than $\mathrm{GL}(1 \mid 1, \mathbb{C})$ there will be *more* of these conditions than the ones in (5). In fact, one obtains (5) plus

$$t_{+z}{}^{z^*} = t_{+-}{}^z = 0. \tag{6}$$

As before, we tacitly include the complex-conjugate equations $t_{-z}{}^{z^*} = 0$, etc., in this list. Since $\mathcal{D}$ satisfies the nondegeneracy condition mentioned below (3), half of the conditions in each of (5)–(6) are actually redundant: They are related to the others by Jacobi identities.

To compare the constraints (5), (6) with (1) we first note that from the definitions

$$t_{AB}{}^C = -T_{AB}{}^C + 2\phi_{[A}M_{B]}{}^C. \tag{7}$$

Thus, of the seven independent complex conditions in (1), two come from the two independent conditions in (5) (Ref. 13) and one more comes from (6).

The remaining four conditions in (1) merely serve to fix a convenient gauge and a unique SO(2) connection, as follows. If D is a vector field in $\mathcal{D}$, we do not necessarily have $[D,D] = 0$, since D is odd. In fact, we require this bracket to be nowhere vanishing, since as mentioned, we want $[D,D]$ and D to span all of $T^{1,0}\hat{X}$.[16] Thus by a normalization choice we can restrict our attention to only those frames for which

$$[E_+, E_+] = 2E_z + fE_+ \tag{8}$$

for some function f on $\hat{X}$. For example, the standard frame $\hat{E}_A$ on $\mathbb{C}^{1|1}$ is normalized in this way [Eq. (3)]. With the choice (8) the structure group is further reduced from G_1 to the subgroup $G \subset G_1$ consisting of U

with $\Delta = 0$, $A = B^2$. One also obtains an additional condition on $t_{AB}{}^C$, namely

$$t_{++}{}^z = 2. \tag{9}$$

Unlike the integrability conditions (5) and (6), (9) is just a gauge condition which sometimes simplifies formulas.

We can further simplify E_A by another gauge choice analogous to (8): We require that

$$2t_{+-}{}^- + t_{++}{}^+ \equiv 0. \tag{10}$$

It is easy to show that given any E_A this condition can always be arranged by a suitable transformation in G without encountering any global obstruction.[5] Also, the residual symmetry group after (10) is imposed is precisely the group of super Weyl and U(1) transformations studied in Ref. 7.

Given a frame e_a on an ordinary Riemann surface, we can define a metric by declaring e_a to be orthonormal, thus reducing the structure group from $\mathrm{GL}(1,\mathbb{C})$ to a residual U(1). Demanding further that $T_{zz^*}{}^{z^*} \equiv 0$ fixes a U(1) connection by (7). Similarly in the super case we fix a unique connection ϕ_z, ϕ_+ by imposing the two conditions

$$T_{zz^*}{}^{z^*} = T_{++}{}^+ = 0. \tag{11}$$

Together with (9) and (10) these account for the rest of the constraints.

In fact, it is already well known that constraints like (11) are "conventional," since they serve only to fix a connection.[17] Moreover it has been shown that in two dimensions *all* the torsion constraints are "conventional" in the sense that given an arbitrary frame and connection ∇_A one can construct another $\tilde{\nabla}_A$ satisfying all the constraints.[18] We would like to distinguish the two uses of the word conventional in the previous sentences. We will call (11) *inessential* constraints because they serve only to fix a connection, *and* the connection never enters into the string action (see below). Similarly (9)–(10) are inessential because they fix part of the gauge group G_1, while the action turns out to be G_1 invariant. On the other hand (5)–(6) are essential because they enforce properties of the frame which are gauge invariant and do enter the action. One can view the result in Ref. 18 as providing a projection from the space of all frames to the integrable ones, but not as a statement that integrability is unnecessary.

The torsion constraints of two-dimensional supergravity thus fall into three classes with distinct geometrical meanings: (5) and (6) follow from the integrability of a reduction of the structure group of the manifold $\hat{X}$ from $\mathrm{GL}(1 \mid 1, \mathbb{C})$ to the group G_1. The other conditions describe a particular gauge choice and serve to fix a connection. Of these constraints only the integrability conditions are the important ones, just as on an ordinary Riemann surface. In the latter case we mentioned how

the conformal structure alone was enough to define an intrinsic operator ∂ and hence the action. Given a frame e^a one can choose a coordinate u conformal for this frame and let $\partial = du\, e_u{}^z \partial_z$; the connection never enters. Similarly, a superconformal structure alone suffices to define the differential operator

$$\hat{\partial} = (du\, d\theta)\det \begin{bmatrix} E_u{}^z & E_u{}^+ \\ E_\theta{}^z & E_\theta{}^+ \end{bmatrix} \partial_+. \qquad (12)$$

For convenience we have chosen the normalization (9), but this is not essential. In (12) $du\,d\theta$ is the Berezin volume form, det is the Berezin determinant, and ∂_+ is E_+ regarded as a differential operator. One can readily show that $\hat{\partial}$ is invariant under arbitrary holomorphic changes of coordinates and under local G transformations of E_A. Thus $\hat{\partial} x^\mu \hat{\partial}^* x^\mu$ is once again a volume form on $\hat{X}$ (at least when X is compact[19]). In fact, $\hat{\partial}$ is essentially the operator introduced in Ref. 7, presented in a way which makes clear its invariance under all of G, not just the super-Weyl$\times$U(1) subgroup. It relies on the first kind of torsion constraints but not on the others.

There are advantages in writing $\hat{\partial}$ in this way. First, (12) makes clear all of the gauge symmetries of the string action. Also, since some of the constraints (1) are not essential, we can save ourselves the trouble of solving the full set of constraints explicitly.

On an ordinary two-dimensional complex manifold none of this would have worked: Even given a distribution $\mathcal{D}$, a formula similar to (12) would fail to be G invariant. The beautiful fact about SRS which makes (12) work is that in the Berezin integral $d\theta$ transforms as the *inverse* of a spinor. Thus it is possible for $du\,d\theta \det(\cdots)$ to cancel the G transformation of E_+, just as $du\,e_u{}^z$ canceled the transformation of ∂_z above.

Supermoduli space now consists of superconformal structures modulo diffeomorphisms. It is smaller than the space of complex structures modulo diffeomorphisms, since one has the additional constraints (6). However, any family of inequivalent SRS remains nontrivial when regarded as merely a family of complex manifolds, as one sees by analyzing the respective deformation problems.[5] Thus supermoduli space sits inside the space of complex structures. From this we can conclude that $\hat{\partial}^*$ varies holomorphically to all orders in moduli, as in the ordinary case,[1] a fact which is important for holomorphic factorization. We will study supermoduli space in greater detail in Ref. 5.

One can think of super Riemann surfaces as uncomfortably poised between one and two dimensions. We cannot combine the coordinates u and θ of $\hat{X}$ into a single coordinate, the way we combine the two real coordinates y^i of X into u. We can, however, define a canonical holomorphic line bundle on $\hat{X}$; these are the half-volume forms above. Remarkably, we can also define a

differential operator $\hat{\partial}$ with values in this bundle. This fact makes possible the definition of the superstring action and first-order systems.[3] It also allows us to generalize the machinery of determinant bundles and its relation to algebraic geometry from the bosonic string to the fermionic case.

[1]See, e.g., P. Nelson, Phys. Rep. **149**, 304 (1987).

[2]We will sometimes omit the prefix "super" when it is clear from context.

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[9]See, e.g., G. Moore, P. Nelson, and J. Polchinski, Phys. Lett. **169B**, 47 (1986); E. D'Hoker and D. Phong, Nucl. Phys. **B278**, 225 (1986).

[10]Throughout we are thinking of supermanifolds in the framework of, e.g., D. Leites, Usp. Mat Nauk **35**, 3, (1980) [Russ. Math. Surv. **35**, 1 (1980); and also M. Rothstein, Proc. Am. Math. Soc. **95**, 255 (1985). For example, we are implicitly always considering *families* of SRS. However, we will make no use here of the details of this framework, other than that $\hat{X}$ is noncompact in its odd directions.

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