Perfect Wave-Number Selection and Drifting Patterns in Ramped Taylor Vortex Flow

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(Received 6 July 1987)

The wave-number band accessible to a homogeneous pattern-forming system collapses to a single wave number if the control parameters are ramped slowly in space from subcritical to supercritical. This selection mechanism is investigated for Taylor vortex flow. Quantitative agreement is found with recent experiments. By variation of both radii independently, any wave number within the stable band can be selected. In addition, there exist ramps which do not permit any static patterns but force them to drift. The drift velocity is calculated. Such a drift is not automatically induced by a large-scale flow.

PACS numbers: 47.10.+g

The problem of pattern selection in pattern-forming systems far from equilibrium has received quite some interest. In particular, the question whether there is a general principle that would single out a unique "preferred state" or a "preferred wave number" has been quite intriguing. This problem has been investigated in various respects: for homogeneous systems,[1-3] and with the inclusion of the effect of boundary conditions,[4] the influence of defects,[5] and the generation of patterns by a propagating front.[6] Since the preferred wave number is expected to be taken on if the structure is allowed to expand or contract itself freely without being hindered by boundaries, Kramer *et al.*[7] replaced one boundary by a subcritical ramp where the control parameters are slowly space dependent and become subcritical far from the homogeneous region. Using a simple one-dimensional reaction-diffusion model as an example, they showed that in this case the wave-number band in fact collapses to a *single* wave number. Some aspects of this selection were also analyzed by Pomeau and Zaleski.[8] Subsequently, this wave-number selection has been investigated by various authors, both theoretically[9-12] and experimentally[3,13,14] for convection[9-11,14] as well as for Taylor vortex flow.[3,9,12,13]

So far, for no system has a theoretical analysis been performed which allows a *quantitative* comparison with existing experiments. Since the wave-number measurements in Taylor vortex flow have proven to be very precise[2,3] this flow lends itself to a detailed test of the approach by Kramer *et al.* which relies heavily on a phase-diffusion equation.[15] Since ramps generally induce large-scale shear flows, the result will in addition test the phase-diffusion concept in the presence of such flows. Moreover, the result will show explicitly that in nonequilibrium systems this selected wave number is *nonuniversal*, i.e., it depends on the way the control parameters are varied. In particular, it is generally different from that selected by propagating fronts.[6]

Here we derive the phase-diffusion equation appropriate for ramps in axisymmetric Taylor vortex flow and solve it for the situations of interest. This equation not only describes the wave-number changes due to the change in the control parameters but also allows for a slow drifting of the pattern. A typical setup is shown in Fig. 1. The ramped part consists of two tapered cylinders with radii $\tilde{R}_1(\tilde{z})$ and $\tilde{R}_2(\tilde{z})$, respectively. In the experiments the homogeneous supercritical part, where the inner radius is taken to be $\bar{R}_1$, is used to measure the wavelength. The two control parameters $\tilde{R}_1$ and $\tilde{R}_2$ are assumed to vary on a slow scale $X=\alpha x=\alpha\tilde{z}/\bar{R}_1$, $\alpha \ll 1$. A dimensionless t is introduced via $t=\tilde{t}\nu/\bar{R}_1^2$. Following the method described in Ref. 7, the incompressible Navier-Stokes equations are expanded in α allowing also the wave number $\tilde{q}=q/\tilde{d}$ to vary slowly in space and time ($\tilde{d}=\tilde{R}_1-\tilde{R}_2$ is the local gap width). Details of this calculation as well as of the numerics used to evaluate the coefficients will be presented elsewhere.[16] Suffice it to say that the oblique solution domain is transformed to a strip with the new coordinate system being locally orthogonal to order α. At lowest order in the expansion in α the equations for the periodic flow with local wave number $q(X)$ are regained. To solve the equations at order α a solvability condition has to be satisfied which is obtained by projection of the equations at order α onto the left zero eigenvector of the linearized operator.[7,10] This gives the phase-diffusion equation,

$$A(q)\partial_t\phi=\partial_x qB(q)+\partial_x \mathcal{T}C(q)+\partial_x \eta D(q), \qquad (1)$$

where the phase ϕ is given by $q=\partial_x\phi$. The flow is 2π periodic in ϕ. Instead of the control parameters $\tilde{R}_1(x)$ and $\tilde{R}_2(x)$, the local Taylor number

$$\mathcal{T}(x)=2\tilde{\Omega}_1^2\tilde{d}^4(\eta^2-\mu)/(1-\eta^2)\nu^2$$

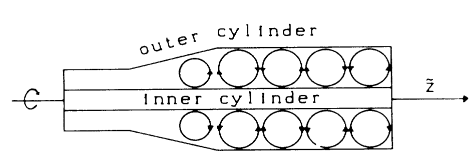

FIG. 1. Schematic setup with ramp on outer cylinder. The ramp becomes subcritical at the left end.

and the radius ratio $\eta(x)=\tilde{R}_1(x)/\tilde{R}_2(x)$ are used to describe the tapering of the cylinders. Here μ is the ratio of the outer to the inner rotation rates $\tilde{\Omega}_2$ and $\tilde{\Omega}_1$, respectively, and ν is the kinematic viscosity. The coefficients A, B, C, and D are functionals of the periodic flow and therefore depend on q. They are too complicated to be given here explicitly. In fact, all analytical calculations in this work were done with an algebraic manipulation program. Before presenting the quantitative results for the solutions of (1) we discuss some qualitative properties of this equation in the static case, $\partial_t\phi=0$, which apply also to ramps in other systems.

(1) All ramps which can be transformed into each other by (nonlinear) coordinate transformations are equivalent, since the coefficients A, B, C, and D are coordinate independent. In particular, it is not the derivatives $\tilde{r}_1'=\partial_x\tilde{R}_1$ and $\tilde{r}_2'=\partial_x\tilde{R}_2$ themselves that are important but only their (local) ratio.

(2) If q is given at one point, e.g., the critical point $(\mathcal{T}=\mathcal{T}_c, q=q_c)$, the phase-diffusion equation fixes q in the whole system. This leads to the well-known perfect wave-number selection by subcritical ramps.[7]

(3) In general, the phase-diffusion equation is not a total differential. Therefore the selected wave number at a given point depends not only on the values of $\mathcal{T}$ and η at this point but also on the path in $(\mathcal{T},\eta)$ space which connects this point with the subcritical region. This *nonuniversality* of the selected wave number will be discussed below in detail.

We first present the results for the perfect selection of the wave number by a subcritical ramp consisting of a stationary conical outer cylinder and a straight inner rotating cylinder ($\mu=0$, see Fig. 1). For a given value $\epsilon_R^h=(\mathcal{T}^h/\mathcal{T}_c)^{1/2}-1$ of the reduced control parameter in the homogeneous section, (1) is integrated starting from a point where $\epsilon_R(x)$ is very small ($\epsilon_R=0.02$) with a wave number $q_{\rm in}$ very close to q_c $(3.1<q<3.15)$. The precise value of $q_{\rm in}$ is essentially irrelevant because with increasing values of ϵ_R^h its influence on the wave number q^h obtained in the homogeneous section decreases very rapidly (see Fig. 3 below). The selected wave number q^h as a function of ϵ_R^h is shown in Fig. 2 for the radius ratio $\eta^h=0.75$ in the homogeneous section. Also shown are the experimental results obtained by Dominguez-Lerma, Cannell, and Ahlers[3] for two different ramping angles δ. They find small wave-number bands which shrink with decreasing ramping angle. As the present calculation applies to the limit $\delta\rightarrow 0$, the agreement is seen to be very good. Thus the phase-diffusion approach is very well suited to the description of wave-number selection by ramps. In order to obtain also the width of the remaining bands for finite δ one would have to apply the theory presented by Riecke[17] to treat the effect of pinning centers like the abrupt transition from the ramp to the homogeneous section present in the experiments (corner). For a model system this theory has been shown[18] to describe the essential qualitative features of the bands exhibited in Fig. 2. Smoothing out the corner should lead to a significant reduction of the wave-number band.[12,17] Therefore we expect the good agreement to persist then also for larger values of δ. These steeper ramps induce much stronger shear flows. The phase-diffusion equation (1) includes the effects of such ramp-induced shear flows to order α through the terms involving $\partial_x\mathcal{T}$ and $\partial_x\eta$.

It has been indicated above that (1) will in general not be a total differential. This is demonstrated in Fig. 3. It shows the selected wave number q^h as a function of

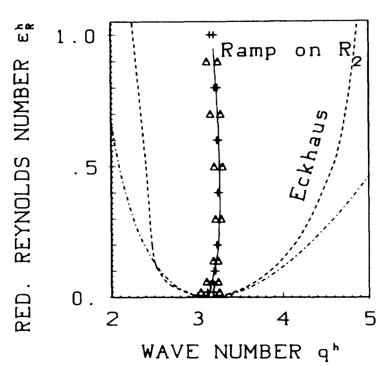

FIG. 2. Wave number q^h in the homogeneous region selected by ramping the outer cylinder (cf. Fig. 1). Solid curve, phase-diffusion equation (1); crosses and triangles, limit of the bands observed in experiments for $\delta=0.0152$ and $\delta=0.0304$ (Ref. 3); dashed curve, Eckhaus stability limit (Ref. 2); dash-dotted curve, Eckhaus stability limit in amplitude approximation.

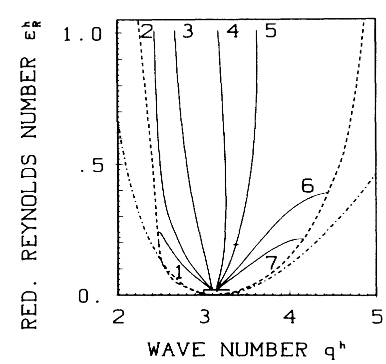

FIG. 3. Different subcritical ramps select different wave numbers q^h. $\tilde{r}_1'/\tilde{r}_2'=$(curve 1) $-1/-0.2$, (curve 2) $-1/0$, (curve 3) $-1/1$, (curve 4) $0/1$, (curve 5) $0.2/1$, (curve 6) $0.4/1$, and (curve 7) $0.5/1$. The bar on curve 5 at $\epsilon_R^h=0.19$ indicates the influence of the initial values of $q_{\rm in}$ (bar at $\epsilon_R^h=0.02$) on the result of (1). Dashed and dash-dotted curves as in Fig. 2.

ϵ_R^h—similar to Fig. 2—for various different linear ramps,

$$\tilde{R}_i = \Theta(-x)\tilde{r}_i' x + \tilde{R}_i^h, \quad i=1,2, \tag{2}$$

with $\Theta(x)$ the unit step function. Clearly, by a suitable choice of the ratio $\tilde{r}_1'/\tilde{r}_2'$ any wave number within the stable band can be obtained. For $\tilde{r}_1'/\tilde{r}_2' = 1/0.82$, the curve for the selected wave number starts horizontally at the critical point.

What happens if the ramp does not end when the wave number reaches the Eckhaus boundary q_E where the pattern becomes unstable with respect to localized phase-slip processes?[2,3] Except for small regions where these processes occur, the system can still be described by the phase-diffusion equation, if the phase-slip centers are considered as sources or sinks of phase. This is because the space and time scales of these processes are small compared to those of the ramp. For the ramp considered here it can be seen that after the decay of transients phase-slip processes occur only at the boundaries.[16] Conservation of phase requires that for stationary solutions the local oscillation frequency ω be constant in space (see also Ref. 10),

$$\partial_t \phi = \omega = U(x) q(x) = \text{const.} \tag{3}$$

These solutions correspond to a perpetual drift of the pattern with velocity $U(x)$. At a phase-slip center one has $q = q_E$. Thus for stationary solutions of (1), one obtains a boundary-value problem which determines the drift velocity $U(x)$. This is in contrast to the description given by Brand[19] who used symmetry arguments to write down a phase-diffusion equation for the case of two conical cylinders with constant gap width, which is not consistent with the phase-diffusion equation derived here. He claims that a second equation for U was necessary to obtain a closed system of equations. This is not possible, as the coefficients in (1) are uniquely determined already by the geometry and the pattern with local wave number q. As just discussed, it is not necessary either if the boundary conditions are taken into account. The connection between stability and drift is illustrated in Fig. 4 for a sample geometry without homogeneous parts. Here the local wave number $q(x)$ together with the local reduced control parameter $\epsilon_R(x)$ is given for a ramp of fixed length $\tilde{L}\bar{R}_1 = 13$. The slopes are taken to be $\tilde{r}_1' = 0.005\bar{R}_1$ and $\tilde{r}_2' = 0.01\bar{R}_1$. The sequence of figures corresponds to the same apparatus driven at three different rotation rates. In Fig. 4(a) the left end of the apparatus is subcritical and the wave number $q(x)$ starts at x_{cr}, $\epsilon_R(x_{cr}) = 0$, with q_c. For this system length $q(x)$ does not reach the Eckhaus limit and no phase-slip center can appear. Therefore the pattern is stationary in spite of the shear flow acting on the pattern. With increasing $\tilde{\Omega}_1$, the length of the supercritical region grows. For some $\tilde{\Omega}_{1l}$, the pattern becomes unstable at the right boundary and leads to the appearance of a phase-slip center, and the drift sets in. Beyond $\tilde{\Omega}_{1l}$ the whole system is supercritical and we have the situation shown in Fig. 4(b). A static solution for this case is shown as the dotted line. This solution does not exist in the whole system: The wave number $q(x)$ becomes singular, $(q - q_E)^2 \propto x_E - x$, since the phase-diffusion constant B/A vanishes at q_E. With $\omega = 0.122$, $q(x)$ is given by the solid line. For still larger values of $\tilde{\Omega}_1$, the Eckhaus band becomes wider and less drift is required to satisfy the boundary condition on q. Eventually above $\tilde{\Omega}_{1u}$ even static solutions stay within the band as shown in Fig. 4(c). Detailed calculations of the drift velocity in dependence of the rotation rate will be presented elsewhere.[16] Drifting patterns like these which come to a halt for large rotation rates have been observed by Wimmer[20] using two conical cylinders with constant gap width corresponding to $\tilde{r}_1' = \tilde{r}_2'$.[21]

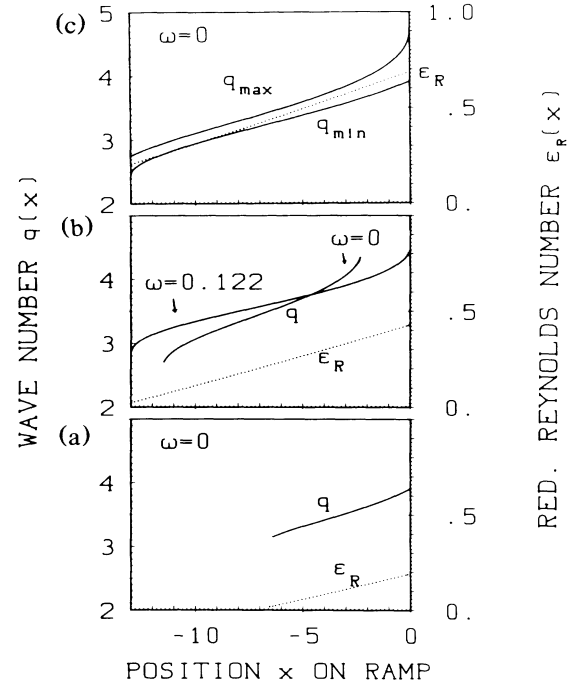

FIG. 4. Dependence of the wave number on the position on the ramp for $\tilde{r}_1' = 0.005\bar{R}_1$ and $\tilde{r}_2' = 0.01\bar{R}_1$ (curve 7 in Fig. 3) for three different rotation rates and $\eta^h = 0.75$ (see also text). Solid curve, $q(x)$; dotted curve, $\epsilon_R(x)$. (a) $\epsilon_R^h = 0.19$. Ramp becomes subcritical at $x = -1.2$. Static pattern with perfectly selected wave number $q^h = 3.9$. (b) $\epsilon_R^h = 0.42$. Ramp stays supercritical. No static pattern exists: $q(x)$ would become singular (at $x = -2.2$ and $x = -0.4$). Pattern drifts with $\omega = 0.122$. (c) $\epsilon_R^h = 0.68$. Ramp stays supercritical. Static pattern exists within a wave-number band $3.8 \le q^h \le 4.7$.

Drifting patterns can also be obtained if two ramps which select different wave numbers are attached to the ends of a homogeneous section.[10,14] This will be discussed in detail separately.[16] In the Taylor system, drifting by incompatible ramps can even be achieved by *monotonic linear* ramps which become subcritical at two distinct points (e.g., for $\tilde{r}_1'/\tilde{r}_2' = 1/0.4$).

It has been shown for the first time that the phase-diffusion theory for systems with space-dependent control parameters describes the experiments on wave-number selection by subcritical ramps quantitatively, even in the presence of a large-scale shear flow. No additional equation is required to describe the influence of this flow: The phase diffusion equation—together with appropriate boundary conditions—is self-contained. In the case of drifting patterns qualitative agreement is found with the experiments performed so far. For a quantitative comparison additional experiments are required, which are being prepared.[22]

We would like to thank G. Ahlers and M Wimmer for sharing their experimental results prior to publication, I. Rehberg, R. H. Busse, and L. Kramer for various stimulating discussions, and the computer center of the University of Karlsruhe for providing free computing time.

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