Local Quantum Field Theory of Possible Violation of the Pauli Principle

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We generalize to a local relativistic quantum field theory a proposal of Ignatiev and Kuzmin for a single oscillator which has small violation of the Pauli principle and thus provide a theoretical framework which, for the first time, allows quantitative tests of the Pauli principle. Our theory provides a continuous interpolation between fully hindered parafermi statistics of order 2 ($\beta=0$), which is equivalent to Fermi statistics, and ordinary parafermi statistics of order 2 ($\beta=1$). We suggest two types of experiments which can place bounds on β.

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Green[1] gave a quantum field theory of particles which obey neither Fermi nor Bose statistics but rather obey a discrete set of parafermi or parabose statistics of integer order $p>1$. The case $p=1$ is Fermi or Bose statistics. Parafermions and parabosons are easily distinguished experimentally from fermions and bosons since integers greater than one are easily distinguished from one. Up to now there has not been a formalism which allows discussion of small violations of Fermi or Bose statistics. Two types of theoretical results have prevented this. At the level of quantum mechanics, Messiah and Greenberg[2] proved that there is an absolute selection rule which forbids transitions between states which contain any number of bosons and fermions and at most one particle which is neither a boson nor a fermion and states which have more than one non-Bose or non-Fermi particle, even when the number of particles is not conserved. They also proved that, in general, the symmetry type of a state of identical particles is absolutely preserved. Independent of Ref. 2, but using the same theorems, Amado and Primakoff[3] specifically criticized attempts[4] to infer bounds on violations of the Pauli principle for electrons from the absence of K-shell x rays from atoms or for nucleons from the absence of γ rays from nuclei, since these attempts to deduce bounds on the validity of the Pauli principle were based on the possibility of transitions between states obeying the Pauli principle and states not obeying it, while such transitions are absolutely forbidden since the Hamiltonian for identical particles must be totally symmetric in their coordinates and thus the symmetry type of the states is conserved by a superselection rule. On the quantum-field-theory level, Greenberg and Messiah[5] proved selection rules, stated below, which prohibit transitions between states having only normal particles and states having only one parastatistics particle in addition to any number of normal particles.

Recently, Ignatiev and Kuzmin (IK)[6] have constructed a model of a single oscillator which is mainly Fermi, but, with small amplitude β, can have double occupancy. As a model of a single oscillator, the IK idea cannot be used to analyze experimental data. In order to make the IK idea useful phenomenologically, we generalize it to a local quantum field theory. We also point out how our theory evades the arguments against the mixing of different types of statistics.

We recall Green's parastatistics which uses the trilinear (anti)commutation relations

$$[[b_k^\dagger,b_l]_\pm,b_m]_-=-2\delta_{km}b_l, \tag{1a}$$

$$[[b_k,b_l]_\pm,b_m]_-=0, \tag{1b}$$

$$[b_i,b_j]_\pm=b_ib_j\pm b_jb_i,$$

where upper (lower) signs are for parabose (parafermi) operators. Green gave a representation of operators obeying these rules with "Green's *Ansatz*":

$$b_k=\sum_{\alpha=1}^{p} b_k^{(\alpha)}, \tag{2}$$

where for equal values of α (the "Green index") the operators obey the usual commutation or anticommutation relations, but for different values of the Green index, the operators have abnormal relative commutation relations,

$$[b_k^{(\alpha)},b_l^{(\alpha)\dagger}]_\mp=\delta_{kl}, \tag{3a}$$

$$[b_k^{(\alpha)},b_l^{(\beta)\dagger}]_\pm=0,\quad \alpha\neq\beta. \tag{3b}$$

The number p of values over which the Green index runs is the "order" of the parastatistics. For parabosons of order p at most p particles can be in an antisymmetric state; for parafermions of order p at most p particles can be in a symmetric state (in particular, at most p particles can be in the same state). For $p=1$, parabosons (parafermions) reduce to bosons (fermions). Greenberg and Messiah[5] derived the absolute selection rules for paraparticles. Two different types of selection rules can occur, depending on the interaction terms in the theory. In one case, the paraparticles of each order p are conserved mod 2. In the other, the number of paraparticles of each order is conserved, mod p. There is no case in which a single paraparticle can decay into normal particles, nor can a paraparticle mix (i.e., form a linear superposition) with a normal particle.

Ignatiev and Kuzmin (IK)[6] constructed a model of a single approximately Fermi oscillator defined by

$$a^\dagger|0\rangle=|1\rangle,\quad a|0\rangle=0,$$
$$a^\dagger|1\rangle=\beta|2\rangle,\quad a|1\rangle=|0\rangle, \tag{4}$$
$$a^\dagger|2\rangle=0,\quad a|2\rangle=\beta|1\rangle.$$

IK gave the following trilinear commutation relations for their oscillator:

$$a^2a^\dagger+\beta^2a^\dagger a^2=\beta^2 a, \tag{5a}$$
$$a^2a^\dagger+\beta^4a^\dagger a^2=\beta^2 aa^\dagger a, \tag{5b}$$
$$a^3=0, \tag{5c}$$

plus the Hermitean conjugate relations. Our first observation is that (5a)–(5c) reduce to the Fermi case for $\beta\to 0$ and to the parafermi, order 2, case for $\beta\to 1$.

For $\beta=1$, (5a) and (5b) reduce to $a^2a^\dagger+a^\dagger a^2=a$, $aa^\dagger=a$, which are the single-oscillator commutation relations for an order-2 parafermi oscillator (see p. B1158 of Ref. 5) with the normalization changed by a factor of $1/\sqrt{2}$. For $\beta=0$, (5a) and (5b) imply $[a^2,a^\dagger]_-=0$, $aa^\dagger a=a$. It is trivial that $[a^2,a]_-=0$, so that if a and $a^\dagger$ are irreducible, which we assume, then $a^2=cI$, where I is the identity, and then $a^2a^\dagger=0$ implies $a^2=0$. These relations are consistent with those of a Fermi oscillator.

For Fock-type representations, the trilinear commutation relations of parastatistics are supplemented by a no-particle and a single-particle condition. The analogs of these conditions for the IK oscillator are

$$a|0\rangle=0,\quad aa^\dagger|0\rangle=|0\rangle. \tag{6}$$

With use of (6), (5a) implies (5b) and (5c) for Fock-type representations. Simply check that (6) and (5a) imply $\|a^{\dagger 3}|0\rangle\|=0$ and also (5b) acting on the no-particle, one-particle, and two-particle states. It is satisfying to see that (5b) is redundant, since commutation relations usually have terms in which the degree of the operator product is reduced.

To see the connection to the parafermi theory, use parafermi operators, b and $b^\dagger$, of order 2, since they will have maximum occupancy two. For a single oscillator, the only terms which can occur are b and $b^\dagger b^2$. A simple calculation shows that the IK oscillator can be expressed with a Green-type *Ansatz,*

$$a^\dagger=\tfrac{1}{2}\sqrt{2}[b^\dagger+\tfrac{1}{2}(\beta-1)b^{\dagger 2}b]$$
$$=\tfrac{1}{2}\sqrt{2}b^\dagger[1+(\beta-1)N_0], \tag{7}$$

$$b=b^{(1)}+b^{(2)},$$
$$N_0=\tfrac{1}{2}[b^\dagger,b]_-+1=b^{(1)\dagger}b^{(1)}+b^{(2)\dagger}b^{(2)}, \tag{8}$$

where $b^{(\alpha)}$ obeys the Green component relations (3a) and (3b). [Since only β^2 appears in (5a) and (5b), β can be replaced by $-\beta$ in (7) and below.] This representation makes it clear that $a^2\to 0$, $\beta\to 0$. Then we see that a is a parafermi operator of order 2 which has been modified by the factor $1-N_0$ which prevents double occupancy and then equipped with the term βN_0 which allows double occupancy proportional to β. Thus Eq. (7) again shows that for $\beta=1$ ($\beta=0$), we can recover a parafermi oscillator of order 2 (an operator which behaves like a Fermi oscillator). As mentioned above, the normalization has also been changed by the factor $1/\sqrt{2}$. We call this construction a "hindered parafermion" or, for short, a "paron" of order 2. We call an electron which obeys the statistics a "paronic electron."

IK give the expression

$$N=A_1a^\dagger a+A_2aa^\dagger+A_3, \tag{9a}$$
$$A_1=(2\beta^2-1)/(\beta^4-\beta^2+1), \tag{9b}$$
$$A_2=(\beta^2-2)/(\beta^4-\beta^2+1)=-A_3, \tag{9c}$$

for the number operator which satisfies $[N,a]_-=-a$. The expressions (7) and (8) allow generalization of the IK oscillator to parons of higher order. We will discuss this elsewhere.

To discuss the field theory based on the IK oscillator, we derive the trilinear commutation relations for an arbitrary set of momentum-dependent operators. We do this in two steps. First, replace the operators a and $a^\dagger$ in (5a) and (5b) by

$$a=\frac{\kappa a_k+\lambda a_l+\mu a_m}{[\kappa^2+\lambda^2+\mu^2+2(\kappa\lambda\delta_{kl}+\lambda\mu\delta_{lm}+\mu\kappa\delta_{mk})]^{1/2}}, \tag{10}$$

and the corresponding formula for $a^\dagger$. Collect all terms with the coefficient $\kappa\lambda\mu$. This gives equations which have terms like $[a_k,a_l]a_m^\dagger$+cyclic permutations. The terms in these equations create or annihilate different net amounts of momentum; thus these equations can be separated into three equivalent equations which are free of the cyclic permutations. The resulting equations are

$$[a_k,a_l]_+a_m^\dagger+\beta^2a_m^\dagger[a_k,a_l]_+ =\beta^2(\delta_{lm}a_k+\delta_{mk}a_l), \tag{11a}$$

$$[a_k,a_l]_+a_m^\dagger+\beta^4a_m^\dagger[a_k,a_l]_+ =\beta^2(a_la_m^\dagger a_l+a_ka_m^\dagger a_l). \tag{11b}$$

(Here and below we use Kronecker deltas as a shorthand for Dirac deltas.) The analogs of (6a) and (6b) are

$$a_k|0\rangle=0, \tag{12a}$$
$$a_ka_l^\dagger|0\rangle=\delta_{k,l}|0\rangle. \tag{12b}$$

Unlike the case of a single oscillator, in which the analogs of (11) and (12) determine all matrix elements of the a's and $a^\dagger$'s, (11) does not determine all matrix elements. We supplement (11) with analogous commutation relations in which the anticommutators are replaced by commutators and unknown parameters are introduced,

$$[a_k,a_l]_-a_m^\dagger = f_1(\delta_{km}a_l - \delta_{lm}a_k) + f_2(a_k a_m^\dagger a_l - a_l a_m^\dagger a_k), \tag{13a}$$

$$a_m^\dagger [a_k,a_l]_- = f_3(\delta_{km}a_l - \delta_{lm}a_k) + f_4(a_k a_m^\dagger a_l - a_l a_m^\dagger a_k). \tag{13b}$$

Equations (13a) and (13b) determine the deviations from the Pauli principle for antisymmetric states while (11a) and (11b) determine the deviations from the Pauli principle for symmetric states. We remark that, in general, higher degree terms might occur in the commutation relations for the commutators. To see if the form (13a) and (13b) suffices, and to determine the unknown parameters, we impose local commutativity on the theory. Specifically, we require that the charge density and the field be relatively local,

$$[\rho(x),\psi(y)]_-|_{x_0=y_0} = -\delta(\mathbf{x}-\mathbf{y})\psi(y). \tag{14}$$

We require that the charge density be bilinear in the spinor field and allow arbitrary coefficients for the two orders of the field,

$$\rho(x) = c_1\psi^\dagger(x)\psi(x) + c_2\psi(x)\psi^\dagger(x). \tag{15}$$

In momentum space, (15) has an integration over momenta. It is easier to work with a relation which is not integrated over momenta, and so we assume the stronger relation,

$$[\rho(x,y),\psi(z)]_-|_{x_0=y_0=z_0} = -\tfrac{1}{2}[\delta(\mathbf{x}-\mathbf{z})\psi(y) + \delta(\mathbf{y}-\mathbf{z})\psi(x)], \tag{16a}$$

$$\rho(x,y) = \tfrac{1}{2}c_1[\psi^\dagger(x)\psi(y) + \psi^\dagger(y)\psi(x)] + \tfrac{1}{2}c_2[\psi(y)\psi^\dagger(x) + \psi(x)\psi^\dagger(y)]. \tag{16b}$$

In momentum space this is

$$[\tfrac{1}{2}c_1(a^\dagger_{-p}a_q + a^\dagger_{-q}a_p) + \tfrac{1}{2}c_2(a_q a^\dagger_{-p} + a_p a^\dagger_{-q}), a_r]_- = -\tfrac{1}{2}[\delta(p+r)a_q + \delta(q+r)a_p]. \tag{17}$$

Now we use (11) to replace anticommutators of the a's and (13) to replace commutators of the $a^\dagger$'s and require that (17) be satisfied identically in terms of the form a and $aa^\dagger a$. These conditions impose four constraints on the six parameters, c_1, c_2, f_i, $i=1$ to 4. The c's are fixed as are two of the four f's. [We note that the $c_{1,2}$ equal the $A_{1,2}$ of (9).] In general, the expression for the left-hand sides of (13a) and (13b) do not have the correct $\beta\to 0$ and $\beta\to 1$ limits; however, the linear combination of (13a) and (13b) which is free of the undetermined f's does have the correct limits. We adopt this as our new commutation relation:

$$(2-\beta^2)[a_k,a_l]_-a_m^\dagger + (1-2\beta^2)a_m^\dagger[a_k,a_l]_- = (1-\beta^2+\beta^4)(\delta_{lm}a_k - \delta_{km}a_l) + 3(1-\beta^2)(a_l a_m^\dagger a_k - a_k a_m^\dagger a_l). \tag{18}$$

Since (5b) was redundant for the single-oscillator case, we consider the possibility that only one linear combination of (11a) and (11b) is necessary. We test this by assuming a commutation relation analogous to (18) for the case involving anticommutators and require that the locality condition (16) or, equivalently (17), still be satisfied. We find that it is satisfied if

$$(2-\beta^2)[a_k,a_l]_+a_m^\dagger - (1-2\beta^2)a_m^\dagger[a_k,a_l]_+ = -(1-\beta^2+\beta^4)(\delta_{km}a_l + \delta_{lm}a_k) + (1+\beta^2)(a_k a_m^\dagger a_l + a_l a_m^\dagger a_k). \tag{19}$$

We take (18) and (19) and their adjoints as the commutation relations for the paronic Fermi fields of order 2. [The commutation relations with only annihilation (creation) operators follow from the commutation relations with both annihilation and creation operators.] Both (18) and (19) are implied by a trilinear commutation relation closely related to (17). To extend the commutation relations to annihilation and creation operators for particles and antiparticles, replace a by a linear combination of b and $d^\dagger$, and $a^\dagger$ by the corresponding adjoint, where the b's and d's are the operators for particles and antiparticles, respectively, in the notation of Bjorken and Drell.[7] The orthonormality properties of the Dirac spinors lead to eight trilinear commutation relations for combinations of annihilation and creation operators for particles and antiparticles. The commutation relations can also be written in position space. We will give details about the commutation relations elsewhere. We emphasize that the choice of these trilinear commutation relations, instead of bilinear commutation relations, for the particle and antiparticle operators suppresses states which are symmetric in particles and antiparticles and extends the violations of the Pauli principle to states containing both particles and antiparticles.

We make explicit the way in which our theory avoids the quantum-mechanics and quantum-field-theory results which absolutely prohibit transitions between normal states and states with just one abnormal particle. Our theory uses a paronic Fermi field of order 2 in which the double occupancy of symmetric states, which is fully allowed for ordinary parafermi fields of order 2, is suppressed by an amount parametrized by β. For $\beta \rightarrow 1$, the ordinary $p=2$ parafermi field is reached; for $\beta \rightarrow 0$, the double occupancy is fully suppressed and the theory is equivalent to a Fermi theory. There is never any "mixing" of different kineds of statistics. We illustrate our theory with electrons assumed to be paronic. If two paronic electrons are brought into contact, then with probability $1-\beta^2/2$ the two-electron state will obey the Pauli principle and with probability $\beta^2/2$ the state will violate the Pauli principle. The states will retain their statistics as long as they remain intact. If an electron is removed from the system and brought into contact with another electron, the new two-electron state will again obey or violate the Pauli principle with probabilities $(1-\beta^2/2)$ and $\beta^2/2$, respectively. Analogous, but more complicated, results hold for many-electron systems.

We suggest two types of experiment to put bounds on β. The probability of finding an atom in which an electron violates the Pauli principle is of order β^2. In stable matter, such electrons would long ago have made transitions to the lowest allowed state; thus we do not expect to observe x rays. Rather such atoms could be detected by exciting them and observing their spectra. It will be difficult to bound β^2 by less than 10^{-8} with spectroscopy. Our second suggestion is to bring slow electrons in contact with an atom and look for x rays coming, with probability β^2, from a transition of an electron in a high Pauli-principle-violating state to a low-lying such state. An efficient way to do this would be to run a high current through a metal and to look for x rays while the current is running. This should give strong bounds on β^2. The old experiments of Ref. 4 are not tests of the Pauli principle; indeed, no high-precision tests of the Pauli principle have been made. Analogous experiments can be made for nucleons in nuclei. We will give further phenomenological analysis elsewhere.

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