Generic $1/f$ Noise in Chaotic Hamiltonian Dynamics

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We describe a mechanism for generic $1/f$ noise in nonintegrable Hamiltonian systems. The phenomenon is observed in the velocity fluctuations of a particle in a 2D periodic potential and is associated with anomalously enhanced deterministic diffusion. It is explained in terms of a renewal process and trapping in a hierarchy of nested cantori.

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Random processes whose power spectral densities $S(\omega)$ diverge like $\omega^{-\alpha}$ with $\alpha \simeq 1$ for $\omega \to 0$ are known as $1/f$ noise.[1] The phenomenon has attracted the attention of physicists for some decades, e.g., because of the pecularity that the random variable has an infinite mean square deviation.[2] Although this might appear unusual, $1/f$ noise is found ubiquitously in various scientific disciplines. It was therefore speculated whether it might originate from a general mathematical mechanism.[1] In particular, with the spread of nonlinear dynamics, many authors have wondered whether this field could bring forth such a mechanism and explain $1/f$ noise as a chaotic phenomenon. The earliest and now best understood mechanism of this type was proposed by Manneville and Pomeau and depends on the existence of a marginally stable fixed point.[3-5] It was observed in Rayleigh-Bénard experiments and Josephson-junction circuits but operates only under special circumstances.[6] It thus cannot account for $1/f$ noise as a generic and typical phenomenon, as it arises, e.g., in metallic films.[1]

In the present Letter we describe a new mechanism that gives rise to $1/f$ noise as a *generic* phenomenon in certain chaotic systems and is closely related to the generic structure of phase space of nonintegrable Hamiltonian systems. We have studied the classical dynamics of a particle moving conservatively in a two-dimensional periodic potential. We have found that the power spectral density $S(\omega)$ of velocity fluctuations typically diverges like $\omega^{-\alpha}$ (with $0.7 \le \alpha \le 1.1$) for diffusive chaotic motions. The diffusion is anomalous, i.e., the mean square displacement of the particle diverges like $t^{1+\alpha}$ for $\alpha < 1$ and like t^2 for $\alpha \ge 1$. Similar to other cases[3-7] this $1/f$ noise is caused by trapping periods in certain regions of phase space. The origin of trapping, however, differs entirely from those cases[3-7] and from excess noise in the Sinai billiard.[8] Here it is the trapping in a self-similar hierarchy of cantori. We give a statistical description in terms of a renewal process and a random walk on a hierarchical lattice. The latter is related to a Markov tree model[9] that was proposed to explain long-time tails in area-preserving maps.[10] It should be pointed out, however, that the Markov tree model alone does not exhibit $1/f$ noise, but on the contrary leads to a vanishing velocity power spectrum at $\omega = 0$, when applied to a map on a compact space. Furthermore we stress that the $1/f$ noise reported here is not a property of a merely mathematical model, but is observed in a physical system. The model of a particle in a periodic potential is a paradigm in solid-state physics. Although an idealization, it arises in various contexts, e.g., for the channeling of ions in crystals,[11] for fast ion conductors (superionic conductors),[12] and for the dynamics of an electron in a crystal when treated classically. Because of the simplicity of the model it is presently not clear, however, whether the observed phenomenon might ultimately be related to $1/f$ noise in metallic films.

We study the dynamics of a classical particle in an analytic two-dimensional periodic potential on a square lattice,

$$H = \tfrac{1}{2} p^2 + V(x,y). \tag{1}$$

For the potential we assume the first terms of a 2D Fourier series for simplicity,

$$V(x,y) = A + B(\cos x + \cos y) + C \cos x \cos y. \tag{2}$$

For numerical calculations we will use the parameters $A = 2.5$, $B = 1.5$, and $C = 0.5$, which represent an egg-carton potential with minima at $V = 0$, saddle points at $V = 2$, and maxima at $V = 6$. It is worthwhile to point out that the deterministic equation of motion following from Eqs. (1) and (2) already exhibits diffusive motions, while in solid-state physics one usually considers random forces as the origin of diffusion.[12] The system is nonintegrable because of the coupling term C. For energies $E \le 2$ the chaotic motion remains confined to a single potential well. For $E > 2$ the particle can move across the saddles from cell to cell.

We have numerically integrated the equations of motion,

$$\ddot{x} = (B + C \cos y) \sin x, \quad \ddot{y} = (B + C \cos x) \sin y, \tag{3}$$

and performed a power spectral analysis of the velocity. Here we focus on a description of the observed $1/f$ noise; more detailed results will be presented elsewhere.[13] Depending on the initial condition the particle carries out

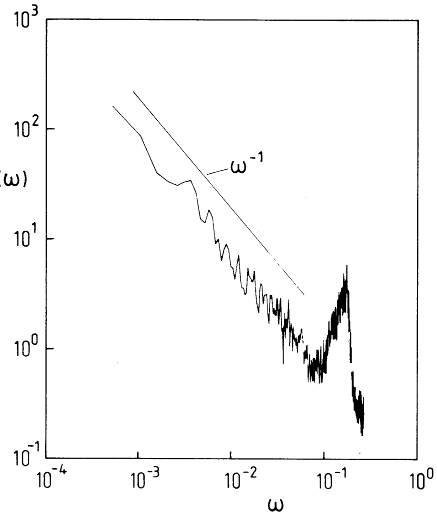

FIG. 1. The velocity power spectral density of deterministic diffusive motions exhibits $1/f$ noise $S(\omega) \sim \omega^{-\alpha}$, here for $E = 4.6$.

periodic and quasiperiodic drift motions and diffusive motions. The latter have a persistent character with long free paths between trappings in a well. We have not found indications of tails in the distribution of trapping times in a cell. The velocity power spectral density defined as

$$S(\omega) = \int_{-\infty}^{\infty} \langle \mathbf{v}(t) \cdot \mathbf{v}(0) \rangle e^{i\omega t} dt \tag{4}$$

was determined with the segment averaging method (80 segments of length 1024). We have found that in a broad energy range (at least $2.5 \leq E \leq 4.6$) the diffusive motions are associated with $1/f$ noise $S(\omega) \sim \omega^{-\alpha}$ with $\alpha \simeq 1$. An example is shown in Fig. 1; the exponent α varies to some extent as shown in Table I. Below $E = 2.5$ spectral analysis is difficult and below $E = 2.0$ diffusion does not exist. Above $E = 4.6$ the spectrum levels off at small frequencies.

In order to understand the origin of the $1/f$ noise we have determined Poincaré surfaces of section at the boundaries of the cells at $x = 2\pi n$ (Fig. 2). Every point in Fig. 2 thus represents a motion of the particle leaving the cell in the perpendicular $\pm x$ direction, and localized motions cannot show up. In the center island there are

TABLE I. Variation of the noise exponent α with the particle energy E.

E	$\alpha(\pm 0.1)$
2.5	0.8
3.0	0.7
4.0	0.75
4.25	1.0
4.35	1.1
4.6	1.0

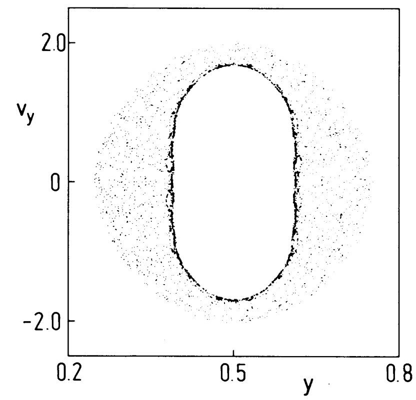

FIG. 2. Poincaré surface of section at the boundaries of the cells ($x = 2\pi n$) for $E = 4.0$. The points represent particle trajectories leaving the cell in the perpendicular direction (free paths). Position is measured in units of 2π.

periodic and quasiperiodic orbits, which are not shown in Fig. 2. They pertain to unlimited free paths (drift motions), as the particle consecutively crosses the edges of the cells. On the other hand, orbits in the chaotic sea surrounding the island only remain there a *finite* time.[14] When they reach its outer boundary, the energy condition to cross the saddle is no longer fulfilled. The free path of the particle thus persists only a finite time and gives rise to diffusion.

Near the inner boundary of the chaotic sea, the orbit seems to have a higher density, which we attribute to the finite observation time. This fact also points to the origin of $1/f$ noise. The orbit can be seen to stick near daughter islands surrounding the central island in Fig. 2. To illustrate this in more detail Fig. 3(a) shows three island chains indicated by representative quasiperiodic orbits. They were isolated by selection of special initial conditions. The magnification in Fig. 3(b) reveals three levels in a hierarchy of daughter islands around daughter islands. We know that generically this hierarchy continues *ad infinitum.*[15] Every island in the chaotic sea is encircled by cantori,[16] partial barriers which the orbit can penetrate. The deeper the orbit enters into the hierarchy of nested cantori, the longer it remains trapped before it can leave the chaotic sea.[14]

We now outline a statistical description of the above mechanism; more details will be found in Ref. 13. We recall that trapping of the orbit in the chaotic sea implies a free path of the particle trajectory. We treat successive free paths as statistically independent and describe their duration T by a probability density $\Psi(T)$. This is justified by the randomizing effect of intermittent localized chaotic motions. The probability $w_0(T)dT$ that at an arbitrary time $t' = 0$ the particle is in a free path of duration T is proportional to T,

$$w_0(T)dT = (T/\langle T \rangle)\Psi(T)dT. \tag{5}$$

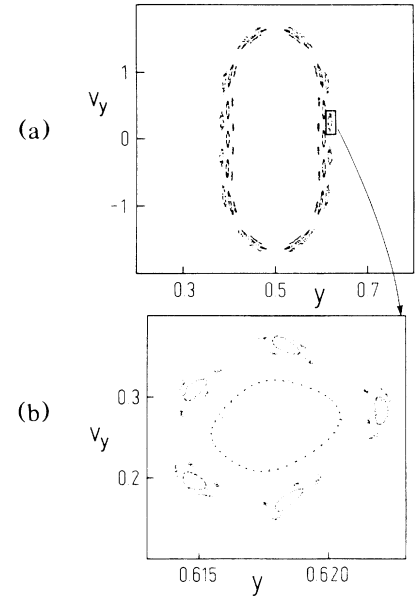

FIG. 3. (a) Isolation of island chains near the boundary of the central island of Fig. 2. (b) Magnification of the box shown in (a), displaying the self-similar hierarchy of daughter islands within the chaotic sea.

The probability that this path persists until time t is the fraction $(T-t)/T$. The probability $w_{0,t}(T)dT$ that the particle is in a free path of length T between time 0 and t is thus

$$w_{0,t}(T)dT = \langle T\rangle^{-1}(T-t)\Psi(T)dT. \tag{6}$$

Approximating the longitudinal velocity by its average v_0 along the paths gives us the velocity autocorrelation function $C(t) = \langle \mathbf{v}(0)\cdot\mathbf{v}(t)\rangle$ as the integral of Eq. (6) over all possible paths:

$$C(t) = \frac{v_0^2}{\langle T\rangle}\int_t^\infty (T-t)\Psi(T)dT. \tag{7}$$

Events consisting of two or more free paths between 0 and t have equal probabilities to end with positive or negative velocity and thus do not contribute to $C(t)$. In terms of the Laplace transforms $\tilde{C}(s)$ and $\tilde{\Psi}(s)$, Eq. (7) turns into

$$\tilde{C}(s) \propto s^{-1} + \langle T\rangle^{-1} s^{-2}[\tilde{\Psi}(s) - 1] \tag{8}$$

from which we easily obtain the spectral density as $S(\omega) = 2\,\mathrm{Re}\,\tilde{C}(\delta - i\omega)$.

In a second step we would like to obtain an expression for $\Psi(T)$. Too little, however, is known with mathematical rigor about the transport across cantori. We are thus forced to make a number of assumptions below. We do know rigorously that generically every island is encircled by cantori,[16] a sequence of which has little flux and acts as barriers.[17] The sequence converges to the boundary circle of the island (Fig. 4), which is believed always to be a critical Kolmogorov-Arnol'd-Moser torus. The latter exhibits scaling properties[18] implying scaling also for the fluxes across the encircling cantori.[17] Embedded between those cantori are the island chains (daughter islands) where the scheme repeats hierarchically on a finer scale (Fig. 4). Let us assume that successive transitions across low-flux cantori can be treated as a Markov process. These transitions then represent a random walk on a hierarchical lattice or Markov tree.[9] Following Ref. 9, we number the area between cantori by a sequence of integers $\mathbf{l} = l_1 l_2 \ldots l_n$, as shown in Fig. 4. We assume a constant branching ratio m and denote by $p_t(\mathbf{l}\,|\,\mathbf{l}')$ the conditional probability that the first transition to $\mathbf{l}'$ occurs at time t, if at time 0 there was a transition to $\mathbf{l}$. The analogous probability for direct transitions (without intermediate states) is denoted by $d_t(\mathbf{l}\,|\,\mathbf{l}')$ and is easily determined from the transition rates. The duration T of a free path (=trapping time in the chaotic sea) is the time of the first transition from states k to the outer state 0, and thus its distribution is

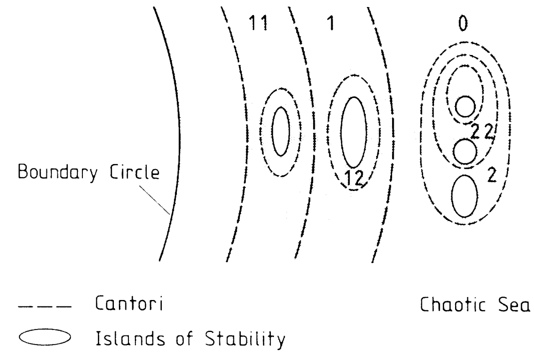

FIG. 4. Schematic representation of the nested hierarchy of cantori. The numbers indicate Markov states.

$$\Psi(T) \propto \sum_{k=1}^{m} p_T(k\,|\,0). \tag{9}$$

The conditional probability must satisfy a set of coupled integral equations:

$$p_t(k\,|\,0) = d_t(k\,|\,0) + \sum_{j=1}^{m}\int_0^t d_\tau(k\,|\,kj)\,p_{t-\tau}(kj\,|\,0)\,d\tau, \tag{10}$$

$$p_t(kj\,|\,0) = \int_0^t p_\tau(kj\,|\,k)\,p_{t-\tau}(k\,|\,0)\,d\tau, \tag{11}$$

which can be solved by Laplace transformation and a scaling *Ansatz* for the functions p_t.[9] This leads to the Laplace transform of $\Psi(T)$,[13]

$$\tilde{\Psi}(s) = 1 + s^{2-\alpha} f(s), \tag{12}$$

where $f(s)$ is an analytic function and the exponent α is determined by the scaling of transition rates. From Eqs. (8) and (12) we obtain the spectral density as $S(\omega)$

$\sim\omega^{-\alpha}$ and thus qualitatively recover the observed $1/f$ noise. Determination of the numerical value of α remains a problem, which is discussed in more detail in Ref. 13. Finally, we discuss how our results are affected by weak dissipation (and external fluctuations). We expect that details of phase space below a certain size are blurred, leading to a low-frequency cutoff in the spectrum. Moreover, one might wonder whether the whole hierarchical organization of phase space breaks down. It was found in other models, however, that the island structure of phase space still shows up under sufficiently weak perturbation.[19]

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[14]Hereby we mean that the particle trajectory turns into a local motion within a well, before it can reach the Poincaré section again.

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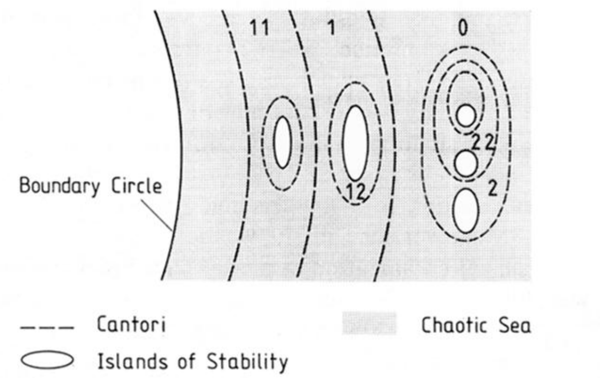

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