Random-Matrix Theory and Universal Statistics for Disordered Quantum Conductors

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An *Ansatz* is proposed for the joint probability distribution of the eigenvalues of the transfer matrix in the quantum transport problem, based on symmetry arguments and a "maximum entropy" hypothesis. The local statistical behavior of the distribution is predicted to be that of the well-known random-matrix ensembles of Wigner and Dyson; and this result is confirmed by independent numerical calculations. For metals this behavior leads to size- and disorder-independent conductance fluctuations, and this approach suggests an alternative framework for the scaling theory of localization.

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The importance of fluctuation effects in the study of localization and quantum transport phenomena has been appreciated for some time. However, early studies[1] focused on the one-dimensional problem in which there is no regime of diffusive (metallic) behavior. Recently the presence of important fluctuation phenomena in the conductance, g, of normal metals has been observed experimentally[2] and studied theoretically.[3,4] The most striking result of these investigations is that the statistical variance of the conductance, averaged over an ensemble of metal samples with the same macroscopic characteristics, satisfies

$$\mathrm{var}(g)\simeq 1, \tag{1}$$

independent of the size of the sample or its average conductance (g is measured in units of e^2/h). Equation (1) predicts much larger fluctuations in g than would be expected from classical reasoning[3] and the correctness of this prediction has been well confirmed experimentally. However, the quantitative arguments for this result have been based on perturbation theory or numerical simulations which have not fully elucidated the fundamental reason for the universality of the conductance fluctuations.

Recently Imry[5] has proposed a more fundamental explanation for Eq. (1) based on the conjecture that the transfer matrix, which determines the conductance, has statistical properties characteristic of the well-known random-matrix ensembles introduced by Wigner and Dyson,[6,7] the Gaussian orthogonal ensemble (GOE) (describing systems with time-reversal symmetry), and the Gaussian unitary ensemble (GUE) (describing systems without). These ensembles have been extremely useful paradigms for studying nuclear energy-level distributions,[8] and more recently they have been widely employed in the study of the quantum mechanics of classically chaotic systems.[9] In this paper we establish the connection of the universal conductance fluctuations (UCF) to random-matrix theory by both analytic and numerical calculations. In addition, we propose that a "maximum entropy" hypothesis (MEH)[7,10] applies to the quantum transport problem, and show that the UCF follow naturally from this *Ansatz*.

We consider an ensemble of disordered conductors of length L_x and cross-sectional area L_y^{d-1} attached to infinite perfectly ordered "leads." The scattering properties of each conductor are completely characterized by a unitary scattering matrix,

$$S=\begin{bmatrix} r & t' \\ t & r' \end{bmatrix}$$

(where r,t, and r',t' are the reflection and transmission matrices from right to left and vice versa), which relates the incident fluxes I,I' to the outgoing fluxes O,O',

$$S\begin{bmatrix} I \\ I' \end{bmatrix}=\begin{bmatrix} O \\ O' \end{bmatrix};$$

or by a transfer matrix, T, which relates the fluxes on one side to those on the other,

$$T\begin{bmatrix} I \\ O \end{bmatrix}=\begin{bmatrix} O' \\ I' \end{bmatrix}.$$

For a fixed energy, E, there will be N propagating channels in each direction, so that S and T are $2N\times 2N$ matrices. The conductance of the disordered region, for $N\gg 1$ and $l/L_x\ll 1$ (l is the elastic scattering length), is given by[3,5] $g=\mathrm{Tr}(t^\dagger t)$. From the definitions and the unitarity of S it follows that

$$g=\mathrm{Tr}(t^\dagger t)=2\mathrm{Tr}\{[T^\dagger T+(T^\dagger T)^{-1}+2]^{-1}\}$$

$$=\sum_{i=1}^{N}\frac{1}{1+X_i}, \tag{2}$$

where $X=\frac{1}{4}[T^\dagger T+(T^\dagger T)^{-1}-2]$, $\{X_i\}$ are the N nondegenerate eigenvalues of this $2N\times 2N$ matrix, and the form of X implies $X_i\geq 0$. Equation (2) hold for each particular conductor, and we thus see that the ensemble-averaged conductance, $\langle g\rangle$, just measures the average number of eigenvalues of X in a small interval near zero, e.g., [0,1]. For a metallic conductor, $\langle g\rangle\gg 1$; thus there must be many X_i in this interval, and this

number must decrease with increasing disorder, approaching unity as the metal-insulator transition. The *variance* of g measures the fluctuations in the number of X_i in the interval [0,1] from conductor to conductor; so that if the X_i were *uncorrelated*, one would expect $\mathrm{var}(g)\simeq\langle g\rangle$, i.e., nonuniversal fluctuations, typically much *larger* than those predicted by Eq. (1). The validity of Eq. (1) thus requires a high degree of correlation in the spectrum of X.

This point of view strongly suggests an analogy between the result of Eq. (1) and the phenomenon of spectral rigidity observed in the spectra of complex nuclei.[8] Briefly, in many complex nuclei it is found that the energy-level sequence is very regular. It deviates very little from a uniformly spaced sequence and this deviation only increases logarithmically with the numbers of levels considered. This behavior implies a long-range correlation of energy levels, and it has been quantitatively understood by the hypothesis that the nuclei exhibit statistical fluctuations characteristic of the GOE and other related ensembles which can be shown to exhibit such long-range eigenvalue correlations.[6-8] In particular, Dyson and Mehta[7] have shown for these ensembles that in an interval L of the spectrum where the density is constant, any linear statistic $Z=\sum_i f(X_i)$ has a variance which is independent of the average number of eigenvalues in L and typically of order unity (f is any smooth function on L, and the sum is only over X_i in L). As Imry[5] noted, *if* this result is valid for the matrix X, it immediately yields Eq. (1), since g is essentially such a statistic (note that the resistance, $1/g$, is *not* a linear statistic). We give analytic arguments and numerical results that confirm this conjecture below.

In our problem the matrix X is determined in a complicated manner from a random Hamiltonian, and we find that, in general, this procedure does not generate one of the standard ensembles. However, applications of random-matrix theory to the nuclear[8] and "quantum chaos"[9] problem have revealed that a much wider class of random-matrix ensembles still exhibit the same fluctuation behavior as the GOE or GUE.[10] The reason for this is that the spectral correlations are typically determined solely by the symmetries of the random matrices, whereas the average spectral density is sensitive to the specifics of a given model. In fact, it has been shown that if an ensemble is as random as possible, given a certain spectral density,[8,10] then its local spectral fluctuations are determined solely by the symmetries of the matrices. The assumption of a "maximum entropy" ensemble has a precise meaning in terms of the minimization of an information functional on the probability distribution of the matrices,[10] $P(X)$. We make such an *Ansatz* for $P(X)$,[11,12] which implies that the joint probability distribution of the eigenvalues, X_i, has the form[10]

$$p(\{X_i\})=J(\{X_i\})\prod_{i=1}^{N}\exp[-f(X_i)], \tag{3}$$

where $f(X_i)$ is a function which is determined from the average spectral density, $\sigma(X)$, by the relation $f(X)\simeq\int dX'\,\sigma(X')\ln|X-X'|$. All the eigenvalue correlations in Eq. (3) are contained in the Jacobian, $J(\{X_i\})$, which is determined solely by the symmetries of the X matrices. We emphasize that $J(\{X_i\})$ is not known, as the symmetries of X do not correspond to any of the previously studied ensembles.[6,8] Thus it is unclear whether the conjectured behavior of the $\{X_i\}$ is consistent with the symmetries of X.

To derive $J(\{X_i\})$, we first consider the matrix $T^\dagger T$ appearing in Eq. (3), whose eigenvalues $\{\lambda_i\}$ are related to $\{X_i\}$ by $X_i=\frac{1}{4}[\lambda_i+\lambda_i^{-1}-2]$. T has $U(N,N)$ symmetry; it preserves the norm $|I|^2-|O|^2$. It is convenient to perform a unitary transformation on T, $P=U_0TU_0$, where

$$U_0=\frac{1}{2}\begin{bmatrix} I & -iI \\ -iI & I \end{bmatrix}$$

(this notation denotes $N\times N$ subblocks of $2N\times 2N$ matrices). The $U,(N,N)$ symmetry of T implies that P is symplectic: $P^\dagger JP=J$ with

$$J=\begin{bmatrix} 0 & -I \\ +I & 0 \end{bmatrix},\quad J^2=-1;$$

this symmetry of P represents current conservation. We treat P and not T, because time-reversal symmetry implies that P is real, whereas T is not, (of course $P^\dagger P$ and $T^\dagger T$ have the same eigenvalues, $\{\lambda_i\}$). Below we outline the derivation for P real, and then indicate the modifications required for a system without time-reversal symmetry. The detailed derivations will be given elsewhere.

Consider $Q=P^{\mathrm{T}}P$, with P a real symplectic $2N\times 2N$ matrix. We need to define a volume element dQ in the matrix space of Q, so that by a change of variables, we can integrate over its eigenvectors, leaving the appropriate Jacobian factor for the eigenvalues, $J(\{X_i\})$. We follow closely Dyson's approach of Ref. 11. Define

$$Q+dQ=P^{\mathrm{T}}(1+dM)P. \tag{4}$$

$\tilde{\rho}(Q)$ and the *independent* elements of dM define a class of measures $\mu(dQ)$ on the space Q, where $\tilde{\rho}(Q)=\prod_i\times\exp[-f(\lambda_i+\lambda_i^{-1})]$ is by the *Ansatz* Eq. (3) independent of the eigenvectors of Q. We can show that these measures are independent of the choice of P, as any matrix P' satisfying (4) must be related to P by an orthogonal transformation. We now compute the independent elements of dM explicitly.

Since $Q=Q^{\mathrm{T}}$, it is diagonalized by a real orthogonal matrix, R, $Q=R^{\mathrm{T}}DR$. Since Q and D are both symplectic, R can be chosen symplectic. Differentiation of the diagonalizing equation for Q gives

$$P^{\mathrm{T}}dM\,P=dQ=dR^{\mathrm{T}}DR+R^{\mathrm{T}}D\,dR+R^{\mathrm{T}}dD\,R.$$

Define $dR=dA\,R$; then $(R+dR)(R^{\mathrm{T}}+dR^{\mathrm{T}})=1$ im-

plies $dA^{T}=-dA$, and $(R+dR)J(R^{T}+dR^{T})=J$ implies $[dA,J]=0$. Thus $P^{T}dM\,P=R^{T}\{D\,dA-dA\,D+dD\}R$, and the crucial point now is that we are free to choose any convenient P to simplify this equation, because of the uniqueness of dM. We choose $P=D^{1/2}R$, giving $dM=D^{1/2}dA\,D^{-1/2}-D^{-1/2}dA\,D^{1/2}+D^{-1}dD$,

$$dM_{ij}=\begin{cases}\lambda_i^{-1}d\lambda_i, & i=j=1,2,\ldots,2N,\\ [(\lambda_i/\lambda_j)^{1/2}-(\lambda_j/\lambda_i)^{1/2}]dA_{ij}, & i\neq j=1,2,\ldots,2N.\end{cases}\tag{5}$$

Since Q is positive symmetric symplectic, its eigenvalues $\{\lambda_i\}$ are real and positive, and occur in inverse pairs. We choose the $\lambda_i>1$ to be independent, and the conditions $dA=-dA^{T}$ and $[dA,J]=0$ leave only N^2 independent dA_{ij} in Eq. (5). Collection of independent terms dM_{ij} yields the measure

$$\mu(dQ)=\prod_{i=1}^{N}d\lambda_i(1-\lambda_i^{-2})\exp[-f(X_i)]\prod_{i<j}^{N}[(\lambda_i+\lambda_i^{-1})-(\lambda_j+\lambda_j^{-1})]dA_{ij}.\tag{6}$$

Changing variables to X_i and integrating over the dA_{ij} yields $J(\{X_i\})$ in Eq. (3):

$$p(\{X_i\})=\textstyle\prod_{i<j}|X_i-X_j|^{\beta}\prod_i\exp[-f(X_i)],\tag{7}$$

where $\beta=1$ (2) for the time-reversal symmetric (asymmetric) case. Without time-reversal symmetry, the previous argument applies with P complex symplectic, R unitary symplectic, and dA anti-Hermitean. This doubles the independent dA_{ij}, and this causes each factor $[(\lambda_i+\lambda_i^{-1})-(\lambda_j+\lambda_j^{-1})]$ to appear twice. Recently Mellow, Pereyra, and Kumar[12] have arrived independently at the same result for the $J(\{X_i\})$, for $\beta=1$, using a different approach.

The crucial new result in Eq. (7) is that the factor in $P(\{X_i\})$ that determines the eigenvalue correlations, $\prod_{i<j}|X_i-X_j|^{\beta}$, has exactly the same form as for a GOE or GUE eigenvalue distribution and thus Imry's conjecture is consistent with the symmetries of X. This means that the MEH, Eq. (3), implies that the fluctuation behavior of $\{X_i\}$ is *identical* to that of the GOE (GUE). Hence, if the *Ansatz* is correct, the UCF are another manifestation of the statistical behavior predicted by random-matrix theory and observed in nuclear and atomic level distributions.

However, Eq. (7) represents only an *Ansatz* for the statistical behavior of the random-matrix ensembles generated by the standard microscopic models describing quantum transport. It is only of great interest *if, in fact, those models generate distributions of this form.* We therefore tested the applicability of the *Ansatz* by comparing its statistical predictions with the behavior of an ensemble of X matrices generated numerically for the Anderson random tight-binding model. To compare our numerical results for the statistics of $\{X_i\}$ with the known statistical properties of the GOE and GUE, we must first correct for a nonnegligible variation in the average spectral density $\sigma(X)$.[9] To do this we use the numerically obtained $\sigma(X)$ to rescale the $[X_i]$ so as to measure their fluctuations in units of the *local* eigenvalue spacing. Equation (7) predicts that after this rescaling the $\{X_i\}$ should exhibit GOE fluctuations in the absence of a magnetic field, and GUE fluctuations in a field. There are two commonly studied fluctuation measures for these distributions[7-9]: the probability density of the nearest-neighbor spacing, $P(S)=a_\beta S^{\beta}\exp(-b_\beta S^2)$ (a_β,b_β are known constants), and the statistic $\Delta_3(L)$ which measures the deviation of a given eigenvalue sequence from a perfectly uniform sequence, as a function of the average number of eigenvalues in the sequence. Thus $\Delta_3(L)$ measures the spectral rigidity crucial for Eq. (1). In Figs. 1 and 2 we show that our numerical results agree perfectly with the GOE and GUE fluctuation behavior predicted by Eq. (7) when $\langle g\rangle\gg 1$. This strongly supports the maximum-entropy hypothesis leading to Eq. (7), and confirms the proposed relationship between random-matrix theory and the universal conductance

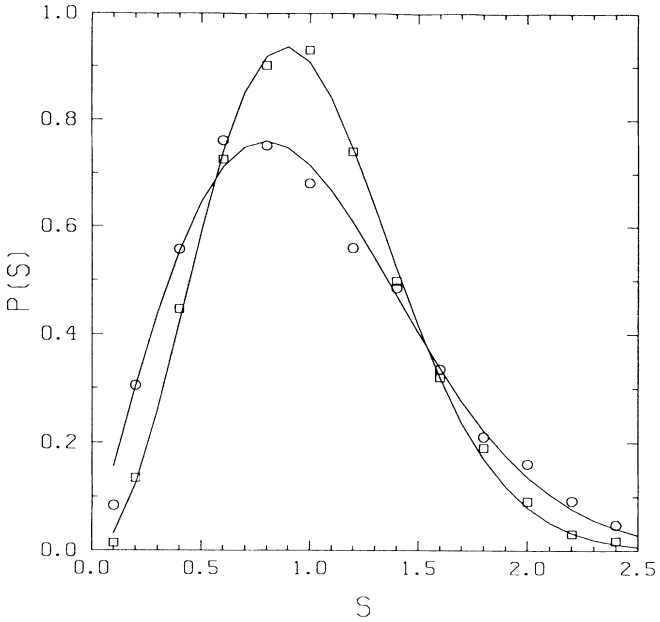

FIG. 1. Probability density of nearest-neighbor eigenvalue spacing for rescaled spectrum of X. Circles are numerical results with no magnetic field, squares with field; solid lines are analytic results for GOE and GUE. Results are for square 2D samples of side 40, disorder parameter $W=1$ ($\langle g\rangle\simeq 10$), ensemble size 400.

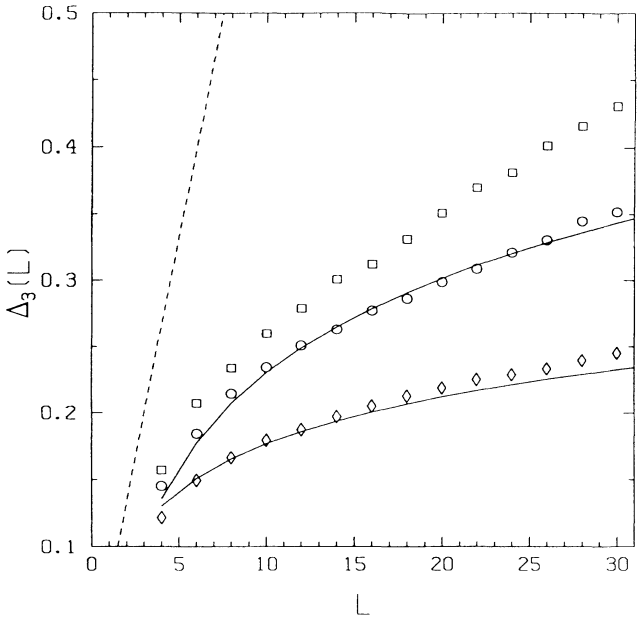

FIG. 2. $\langle\Delta_3(L)\rangle$ as defined in text, determined numerically without a field (circles) or with a field (lozenges) for same parameters as Fig. 1 ($\langle g\rangle\simeq 10$). Solid lines are analytic GOE, GUE results. Squares are data for $\langle g\rangle\simeq 1$ ($W=4$), showing significant deviation as discussed in text. Dashed line is Poisson result $\langle\Delta_3(L)\rangle = L/15$ expected for uncorrelated spectrum.

fluctuations.

For $\langle g\rangle\simeq 1$, when the system is becoming localized, we find significant departures from the GOE result for $\Delta_3(L)$. This raises two possibilities. Either the MEH leading to Eq. (7) may be breaking down as the system localizes, or the numerical rescaling procedure we have used may be less accurate for less dense spectra.

Finally, we briefly comment that Eq. (7) provides a natural framework for the scaling theory of localization, since it factorizes $p(\{X_i\})$, into a universal factor describing eigenvalue fluctuations, and a factor which determines the average spectral density $\sigma(X)$ that contains all information about the system parameters (energy, degree of disorder, sample dimensions). What is needed to establish the validity of one-parameter scaling from this point of view is to show that $\sigma(X)$ is specified by only one parameter for every set of system parameters. If this is so, then $p(\{X_i\})$ is a one-parameter distribution, and since it determines the probability distribution, $W(g)$, the one-parameter scaling hypothesis for $W(g)$ would be proven.[13] Since the one-parameter scaling hypothesis for $W(g)$ has recently been challenged,[14] this question deserves further study.

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