Understanding Electroweak Couplings of the Pion as a $q\bar{q}$ Composite

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Three basic pionic amplitudes f_π^+, F_π (for $\pi^0 \rightarrow \gamma\gamma$), and $F_{3\pi}$ (for $\pi\gamma \rightarrow \pi\pi$) are calculated in a QCD-oriented Bethe-Salpeter model of $q\bar{q}$ hadrons which fits the spectra of all quarkonia. The results, $f_\pi^+=157$ MeV (experiment, 135), $\Gamma(\pi^0 \rightarrow \gamma\gamma)=8.03$ eV (7.9), and $F_{3\pi}=12.7$ GeV^{-3} (13 ± 2), obtained with no free parameters, agree with the data and with the Wess-Zumino-Witten theory.

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While a composite ($q\bar{q}$, qqq) hadron is natural in a QCD picture, the pion seems to stand apart, as testified by the elegance of chiral-anomaly theories[1,2] characterized by a dimensional parameter f_π governing the interactions of an elementary pion field. In particular, the Wess-Zumino-Witten theory[2] is consistent with the low-energy limit of QCD,[3] with testable predictions such as the low-energy theorem[4] $ef_\pi^2F_{3\pi}=F_\pi$, connecting the amplitudes for $\pi \rightarrow l\bar{l}$, $\pi^0 \rightarrow \gamma\gamma$, and $\gamma \rightarrow 3\pi$, as well as other types of predictions.[5] A recent measurement[6] of the $\gamma \rightarrow 3\pi$ amplitude shows good agreement with this prediction, thus apparently vindicating the Wess-Zumino-Witten theory[2] with SU(3) color.[3] On the other hand, the theory in its present "effective" form does not have a mechanism for the pion's response to a high-energy probe which should reveal its $q\bar{q}$ structure through appropriate processes (e.g., $\pi+A \rightarrow \mu\bar{\mu}+X$, $e^+e^- \rightarrow \pi+X,\dots$). Fuller information, all the way from low to high energies, is more naturally contained in the pion wave function.

The purpose of this Letter is to present the results of this alternative "wave-function" point of view for the pion, with respect to quantities f_π, F_π, and $F_{3\pi}$, with use of a QCD-oriented Bethe-Salpeter (BS) model of confinement[7-10] for all $q\bar{q}$ hadrons (pion included). The results are not only consistent with the low-energy theorem noted above, but indeed the calculated value of $F_{3\pi}$ at 12.7 GeV^{-3} and the $\pi^0 \rightarrow 2\gamma$ width ($F_\pi^2M^3/64\pi$) at 8.03 eV are in excellent agreement with the corresponding measured values 13 ± 2[6] and 7.9 eV,[11] respectively, while the quantity $f_\pi^{(+)}=\sqrt{2}f_\pi$ at 157 MeV is only 18% higher than its experimental value of 135 MeV, all with no free parameters beyond the three basic constants of the model (ω_0, C_0, m_q) already determined by the hadron spectra.[10] In these agreements the structure of the pion-quark vertex function (to be described below) has proved crucial. Though not the subject of this paper, we note for completeness that equally good fits have been obtained for the pion structure function[12] (via a Drell-Yan process) and the corresponding fragmentation function[12] (via $e^+e^- \rightarrow \pi+X$) as possible high-energy tests of the same vertex function.

Pion-quark vertex function.— Before presenting the results we summarize the essential details of a BS model for vector confinement with a harmonic-oscillator shape[7] which works on a two-tier basis: (1) three-dimensional (null-plane) formulation[7,10] for making contact[10] with the data on hadron mass spectra and (2) a prior reconstruction of the *four-dimensional* BS wave function[8] for the evaluation of various transition amplitudes. The first step suppresses the role of virtual $q\bar{q}$, etc., effects (alternatively, that of higher Fock-space amplitudes[13]) on the spectral calculations, a procedure justified *a fortiori* from the excellent fits to the data on all quarkonia, light and heavy.[10] The second step restores these effects on the transition amplitudes perturbatively via 4D Feynman diagrams.[8] Since the initial formulation,[7,8] the model has undergone two refinements: First, a Lorentz-invariant generalization of the harmonic kernel now permits a mathematically *covariant* formulation[9] of the 3D BS equation in null-plane variables, the longitudinal component A_3 of any three-vector $\mathbf{A}$ reading as[9] $A_3=A_+M/P_+$ (P_μ=hadron four-momentum), so that (over and above the inclusion of virtual $q\bar{q}$, etc., effects) the reconstructed 4D wave function[9] is now valid for hadrons in arbitrary motion. Secondly, certain physical shortcomings of the earlier formulation[7] have been overcome[10] through (i) the *Ansatz* $\omega_{q\bar{q}}^2=wm_q\omega_0^2\alpha_s$ for the spring constant $\omega_{q\bar{q}}^2$ (ω_0=universal constant for all flavors) and (ii) the introduction of an additive universal constant C_0 (which now gives correct zero-point energies). Details are given in Ref. 10 which finds excellent fits to $q\bar{q}$, $Q\bar{q}$, and $Q\bar{Q}$ spectra with

$$\omega_0=158 \text{ MeV}, \quad C_0=0.296, \quad m_q=270 \text{ MeV}, \tag{1}$$

predicting, among other things, the pion mass at 163.1 MeV. For the present calculation of pionic transition amplitudes the structure of the 4D pion wave function is[9]

$$\Psi(p_1,p_2)=i^2S_F(p_1)\Gamma(\mathbf{q})S_F(-p_2), \tag{2}$$

$$\Gamma(\mathbf{q})=N_-\gamma_5D_+(\mathbf{q})\Phi/2\pi i. \tag{3}$$

The crucial quantity is the normalized $\pi q\bar{q}$ vertex function $\Gamma(\mathbf{q})$, which is a product of the null-plane-approximation denominator function D_+ and wave function Φ, together with an overall BS normalizer N_- calculated[8,9] in the standard manner. Now,

$$D_+ = 2P_+(\mathbf{q}_\perp^2 + m_q^2 - \tfrac{1}{4}M^2) + 2P_- q_+^2, \tag{4}$$

where

$$P = p_1 + p_2,\quad 2q = p_1 - p_2,\quad P_+P_- = M^2.$$

The structure of Φ, and hence of N_-, depends on the on- or off-shell character of the associated quark momenta (p_1, p_2). For both p_1 and p_2 on shell, we have[9,10]

$$\Phi = \exp[-\tfrac{1}{2}\beta^{-2}(\mathbf{q}_\perp^2 + M^2 q_+^2 P_+^{-2})], \tag{5}$$

$$(4\pi^3\beta^2)^{3/4} P_+ N_- = \tfrac{1}{2}M(m_q^2 + \tfrac{3}{2}\beta^2 + \tfrac{1}{4}M^2)^{-1/2}, \tag{6}$$

$$\beta^4 = 2\omega_0^2 m_q M \alpha_s \gamma^{-2},\quad \gamma^2 = 1 + 2\omega_0^2\alpha_s/Mm_q - 4C_0\alpha_s m_q/M, \tag{7}$$

$$\alpha_s = \tfrac{12}{29}\pi[\ln(4m_q^2/\Lambda^2)]^{-1},\quad \Lambda = 250\ \text{MeV}. \tag{8}$$

On the other hand if both p_1 and p_2 are off shell, the quantity $q_+^2 M^2/P_+^2$ in the exponent of (5) must be replaced by[9] $(\mathbf{P}_\perp = 0)$

$$q_+^2 M^2 P_+^{-2} \to q_+^2(m_q^2 + \mathbf{q}_\perp^2)/p_{1+}p_{2+} \tag{9}$$

so that the off-shell wave function $\tilde{\Phi}$ reads

$$\tilde{\Phi} = \exp[-\tfrac{1}{2}\beta^2(\mathbf{q}_\perp^2 + x^2 m_q^2)(1-x^2)^{-1}], \tag{10}$$

$$x = 2q_+ P_+^{-1},$$

which is consistent with the general form of $f(q^2 + m_q^2/(1-x^2))$ employed by other authors.[14,15] The corresponding normalizer $\tilde{N}$ in this model is given by

$$(8\pi^4\beta^2)^{1/2} P_+ \tilde{N}_- = \int_{-1}^{+1} dx(1-x^2)[\beta^2(1-x^2) + m_q^2 + \tfrac{1}{4}M^2(1+x^2)]\exp[-m_q^2 x^2 \beta^{-2}(1-x^2)^{-1}]. \tag{11}$$

$\pi \to l\bar{l}$ and $\pi^0 \to \gamma\gamma$ couplings.— We follow the procedure for calculating the matrix elements for these transition amplitudes,[8] but attuned to the null-plane-approximation covariant vertex function[9] as summarized above. For both these cases [Figs. 1(a)-1(c)] the off-shell quantities $\tilde{\Phi}$ and $\tilde{N}$, Eqs. (10) and (11), are appropriate. The amplitude $f_\pi^{(+)}$, Fig. 1(a), is defined as

$$f_\pi^+ P_\mu = \sqrt{3}i(2\pi)^{-4}\int d^4q\, \mathrm{Tr}(\Psi\gamma_\mu\gamma_5) \tag{12}$$

which, after integration over dq_- in the standard manner,[8,9] yields

$$f_\pi^+ = 2\sqrt{3}m_q\tilde{N}_- P_+ \int d^2q_\perp \int_{-1}^{+1} dx\, \tilde{\Phi} \tag{13}$$

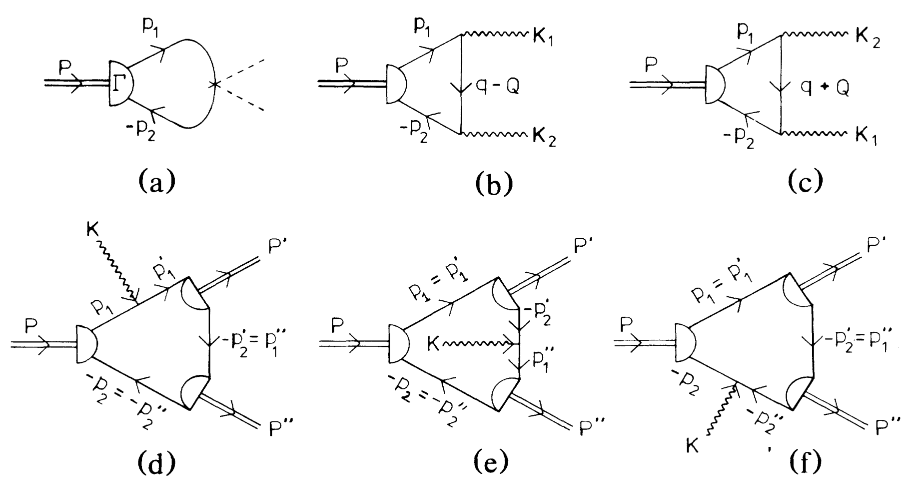

FIG. 1. (a) $\pi \to l\bar{l}$ diagram. (b),(c) $\pi^0 \to \gamma\gamma$ diagrams. (d),(e),(f) $\pi\gamma \to \pi\pi$ diagrams.

leading, after necessary substitutions, to the value

$$f_\pi^+ = \sqrt{2} f_\pi = 157 \text{ MeV}, \tag{14}$$

while experiment gives 135 MeV. Next, the invariant amplitude for $\pi^0 \to \gamma\gamma$ is given by[8]

$$A(\pi^0\gamma\gamma) = 6^{-1/2} e^2 \mathrm{Tr} \int d^4q [\Psi(p_1,p_2) i\gamma\cdot\epsilon^{(1)} s_F(q-Q) i\gamma\cdot\epsilon^{(2)} + (1\to 2)], \tag{15}$$

where $Q = k_1 - k_2$, and the second term interchanges the two photons [Figs. 1(b) and 1(c)]. Its general structure defines the $\gamma\gamma$ form factor F_π of Refs. 4 and 6 through the identification

$$A(\pi^0\gamma\gamma) \equiv F_\pi \epsilon_{\mu\nu\rho\sigma} \epsilon_\mu^{(1)} \epsilon_\nu^{(2)} P_\rho Q_\sigma, \tag{16}$$

$$F_\pi = e^2 (\tfrac{1}{6})^{1/2} \int d^4q \frac{4m_q}{\Delta_1\Delta_2} \frac{D_+\tilde{\Phi}}{2\pi i} \left[\frac{1}{\Delta_3^+} + \frac{1}{\Delta_3^-}\right], \tag{17}$$

$$\Delta_{1,2} = m_q^2 + p_{1,2}^2, \quad \Delta_3^\pm = m_q^2 + (q \mp Q)^2. \tag{18}$$

The QED gauge invariance of (16) is explicit. A straightforward but lengthy integration over dq_- gives rise to the following structure:

$$F_\pi = 4\pi e^2 \tilde{N}_- (\tfrac{1}{6})^{1/2} \int_\lambda^\infty \frac{dt}{t} \left[1 - \frac{\lambda}{t}\right]^{1/2} e^{\lambda - t}, \tag{19}$$

$$\lambda = \tfrac{1}{2} m_q^2 \beta^{-2},$$

which yields $F_\pi = 25.64 \text{ MeV}^{-1}$ and a $\pi^0 \to \gamma\gamma$ width

$$\Gamma(\pi^0\gamma\gamma) = F_\pi^2 M^3 / 64\pi = 8.03 \text{ eV} \tag{20}$$

with use of the observed mass $M = 135$ for π^0, while experiment[11] gives 7.9 eV.

$\gamma \to 3\pi$ coupling.—This process is governed by Figs. 1(d)–1(f) corresponding to the three possible ways in which the photon can interact with the quark lines forming the inner triangle, together with three more diagrams corresponding to the loop direction reversed. These permutations formally ensure QED gauge invariance for the entire $\gamma \to 3\pi$ amplitude in the standard fashion.[16] We use the same procedure for the color and isospin factors as in Ref. 8, and the same notation for the four-momenta of the quarks (p_1, p_1', p_1''), antiquarks (p_2, p_2', p_2''), pions (P, P', P''), and photon (k). The connections are illustrated for Fig. 1(d) as[8]

$$\begin{gathered} p_{1,2} = \tfrac{1}{2} P \pm q, \quad p_{1,2}' = \tfrac{1}{2} P' \pm q', \\ p_{1,2}'' = \tfrac{1}{2} P'' \pm q'', \quad k + p_1 = p_1', p_1'' = -p_2', \\ p_2'' = p_2 = p_2' + P'', \quad P + k = P' + P''. \end{gathered} \tag{21}$$

As to the actual dynamics for the various vertex functions, it is important to recognize that this highly *peripheral* $\gamma\pi \to \pi\pi$ process calls for the *on-shell* form (5) for Φ at *each* $\pi q\bar{q}$ vertex, with the corresponding norm N_-, Eq. (6), to be substituted in Eq. (3) for each pion. The resultant invariant amplitude for Fig. 1(d) is

$$A_1 = (2\pi)^5 i 6^{-3/2} N_- N_-' N_-'' \int d^4q \, D_+ D_+' D_+'' \Phi\Phi'\Phi'' (\mathrm{TR})_1, \tag{22}$$

$$(\mathrm{TR})_1 = \mathrm{Tr}[\gamma_5 S_F(p_1) i\gamma\cdot\epsilon S_F(p_1') \gamma_5 S_F(p_1'') \gamma_5 S_F(-p_2)], \tag{23}$$

with similar expressions for Figs. 1(e) and 1(f), suffixed by indices and 2 and 3, respectively. (Three more terms with the loop direction reversed give merely a factor of 2 for this process.) As in the $\pi^0\gamma\gamma$ case the amplitude[4,6] $F_{3\pi}$ is defined through

$$A(\gamma 3\pi) = 2(A_1 + A_2 + A_3) \equiv F_{3\pi} \epsilon_{\mu\nu\rho\sigma} \epsilon_\mu P_\nu P_\rho' P_\sigma'', \tag{24}$$

where $F_{3\pi}$ can be identified, after a little algebra, as

$$F_{3\pi} = (2\pi)^3 i^{-1} 8 e m_q N_- N_-' N_-'' \int d^4q \, D_+ D_+' D_+'' \Phi\Phi'\Phi'' (\Delta_1\Delta_2)^{-1} [(\Delta_1'\Delta_2')^{-1} + (\Delta_2'\Delta_1'')^{-1} + (\Delta_2''\Delta_1'')^{-1}], \tag{25}$$

with the Δ's defined as in Eq. (18) with four-momenta indicated in Figs. 1(d)–1(f). Again, the QED gauge invariance of Eq. (24) is explicit as expected.[16] The q_--pole integration[8,9] is greatly simplified under the kinematical conditions[6] of the problem which justifies our taking the final pions (p', p'') almost at rest in the c.m. frame of P and k. As a result the overall four-momentum condition (21) simplifies to

$$P_+ = k_0 + k = P_+' + P_+'' = 2M, \quad k_+ = k_0 + (-k) = 0, \tag{26}$$

taking account of the near mass-shell condition $(k_0 = k)$ for the photon. The rest of the procedure is straightforward though lengthy. The final result simplifies to

$$F_{3\pi} = \tfrac{64}{27} e (\pi/\beta^2)^{3/4} (m_q/\sqrt{3}) \exp[-\tfrac{1}{4} M^2 \beta^{-2}] (m_q^2 + \tfrac{3}{2}\beta^2 + \tfrac{1}{4} M^2)^{-3/2} \tag{27}$$

$$= 12.69 \text{ GeV}^{-3} \tag{28}$$

with use again of the values listed in Eq. (1); experiment[6] gives $13 \pm 0.9 \pm 1.3$ GeV^{-3}. We thus have the results on three basic quantities $(f_\pi, F_\pi, F_{3\pi})$ in terms of a composite $(q\bar{q})$ pion model, all in good agreement with the data, but for the possibility of higher-order corrections which might vitiate these agreements.[17] We have checked the effect of second-order gluonic corrections to the $\pi \to \gamma\gamma$ amplitude, Figs. 1(b) and 1(c), in the following manner: There are *three* self-energy corrections, one on each internal quark line; *two* "vertex" insertions at the photon vertices; and *one* at the $\pi q\bar{q}$ vertex. All these corrections are necessary for a consistent treatment of the infrared (ir) divergences inherent in the gluon insertions. If we keep track of the main logarithmic (ir) terms and work entirely in the *Landau gauge,* the resultant correction to the $\pi \to \gamma\gamma$ amplitude, Eq. (15), works out as an overall multiplicative factor:

$$Z_{\rm ir} = 1 + \frac{2\alpha_s}{3\pi}\left[\ln\frac{m_q^2}{\lambda^2} - \frac{m_q^2 - \frac{1}{2}M^2}{m_q^2 - \frac{1}{4}M^2}\ln\frac{m_q^2 - \frac{1}{4}M^2}{\lambda^2}\right], \tag{29}$$

where λ is the ir frequency. Note that the correction vanishes exactly in the limit $M \to 0$ (which corresponds to a point-pion coupling to 2γ via the anomaly term), a result not inconsistent with the Adler-Bardeen theorem[18] (which applies to this limit). For a *finite* pion mass, a natural ir cutoff value is $\lambda = M$, which gives $Z \approx 1 + 0.02\alpha_s$ only, so that our basic result, Eq. (20), remains almost unaffected by such a multiplicative Z^2 correction.

In conclusion, our BS model for $q\bar{q}$ hadrons, apart from giving good fits to the hadron spectra,[10] predicts pionic transition amplitudes consistent with the data[6,11] *and* with the chiral pion limit.[1-4] In addition, this composite model also happens to provide a parameter-free handle on certain high-energy processes probing the pion $q\bar{q}$ structure, again in good agreement with the data.[12] A more detailed report, including the results on a few other allied amplitudes, is in preparation.

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