Quark-Number Susceptibility of High-Temperature QCD

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We measure the response of the quark number to an infinitesimal chemical potential in high-temperature QCD with two light flavors of dynamical fermions. In the chirally symmetric phase the susceptibilities for quark-number density and for the density of the third component of isospin are large and equal within statistics, which is consistent with a plasma of light quarks. In the broken-symmetry phase the susceptibility for quark-number density is small, as expected from quark confinement.

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The behavior of quantum chromodynamics at high temperature is of fundamental and possible experimental interest. Numerical simulations have shown that QCD with light quarks experiences a restoration of chiral symmetry as the temperature is raised,[1-5] with a transition temperature in the range from 100 to 160 MeV.[6] The restoration of chiral symmetry is signaled by the vanishing of $\bar{\psi}\psi$ as the quark mass goes to zero. Beyond this our knowledge of the nature of high-temperature QCD is scant. The high-temperature phase is referred to as the "plasma" phase, and is sometimes pictured as a weakly interacting gas of quarks and gluons. For temperatures slightly above the transition temperature, this picture is clearly simplistic. The QCD coupling is not small at this energy scale. Measurements of the energy and pressure of the plasma show a large energy density in the high-temperature phase similar to a Stefan-Boltzmann law on a lattice.[5] However, slightly above the transition temperature the pressure is small, in contrast to $P=\frac{1}{3}E$ expected for a relativistic gas. It has been suggested[7] that the long-distance behavior of the high-temperature phase is characterized by the propagation of color-singlet objects, just as in the more familiar low-temperature phase.

One available probe of the physics of the plasma is the study of screening lengths.[8,9] Here one measures the propagator from a quark-antiquark or a three-quark source in a spatial direction on the high-temperature lattice. Screening lengths so measured show the parity doubling expected when chiral symmetry is restored. In particular, the nucleon and its opposite-parity partner have equal screening lengths, with inverse screening length comparable to their low-temperature masses.

Here we report a measurement of another set of probes of the plasma, the susceptibility of the quark-number density to changes in the chemical potential. The singlet susceptibility χ_S and nonsinglet susceptibility χ_{NS} are defined as

$$\chi_{S,NS}=(\partial/\partial\mu_u \pm \partial/\partial\mu_d)(n_u \pm n_d). \tag{1}$$

χ_S measures the response of the total quark-number density to change in the chemical potential, while χ_{NS} measures the response of the third component of the isospin density. (We work with two flavors of equal-mass quarks in this paper, but the formalism is easily generalized.) Here n_u and n_d are the expectation values of the number densities of up and down quarks, and μ_u and μ_d are the corresponding chemical potentials.

$$n_{u,d}=(V_s\beta)^{-1}\frac{\partial \ln Z}{\partial\mu_{u,d}}, \tag{2}$$

where Z is the partition function, V_s is the spatial volume, and β is the inverse temperature.

These susceptibilities can be measured in a standard simulation of QCD at zero chemical potential. For the purpose of orientation, consider a gas of free quarks. For low quark masses, both χ_S and χ_{NS} are expected to be large, since it will be relatively easy to create an additional quark or antiquark. For example, in the continuum limit if the quark mass m is much less than the temperature T then $\chi_S \rightarrow N_f T^2$, where N_f is the number of quark flavors. In contrast, if the quarks are massive, then χ_S is suppressed by a factor of $\exp(-m/T)$. As noted above, the plasma is strongly interacting and the excitations are unlikely to be free particles of any mass, but these limiting cases illustrate the nature of this probe of the plasma. In the phase in which the chiral symmetry is broken, χ_S is expected to be small since quarks are confined and the only states with nonzero quark number have large masses. However, if the lowest mass excitation in this phase is an isospin-1 pion, then χ_{NS} will be different from zero, and will increase with decreasing pion mass.

In the continuum, the partition function for two flavors of quarks has the form

$$Z=\int[\delta U]e^{-S_g(U)}\det M_u(U,\mu_u)\det M_d(U,\mu_d). \quad (3)$$

S_g is the action for the pure gauge theory, U is the gauge field, and $M_{u,d}$ are the fermion matrices for the up and down quarks. Then Eq. (2) can be rewritten as

$$n_{u,d}=(V_s\beta)^{-1}\langle\mathrm{Tr}\{M_{u,d}^{-1}\,\partial M_{u,d}/\partial\mu_{u,d}\}\rangle_U. \quad (4)$$

$\langle\ldots\rangle_U$ denotes averaging over the gauge configurations weighted as in Eq. (3). The susceptibilities are then given by

$$\chi_{\mathrm{S,NS}}=\frac{1}{V_s\beta}\left\langle\mathrm{Tr}\left\{\sum_i\left[\frac{1}{M_i}\frac{\partial^2M_i}{\partial\mu_i^2}-\frac{1}{M_i}\frac{\partial M_i}{\partial\mu_i}\frac{1}{M_i}\frac{\partial M_i}{\partial\mu_i}\right]\right\}\right\rangle_U+\frac{1}{V_s\beta}\left\langle\left[\mathrm{Tr}\left\{\frac{1}{M_u}\frac{\partial M_u}{\partial\mu_u}\right\}\pm\mathrm{Tr}\left\{\frac{1}{M_d}\frac{\partial M_d}{\partial\mu_d}\right\}\right]^2\right\rangle_U$$
$$-\frac{1}{V_s\beta}\left\langle\mathrm{Tr}\left\{\frac{1}{M_u}\frac{\partial M_u}{\partial\mu_u}\right\}\pm\mathrm{Tr}\left\{\frac{1}{M_d}\frac{\partial M_d}{\partial\mu_d}\right\}\right\rangle_U^2, \quad (5)$$

where the plus signs are for χ_S and the minus signs for χ_NS. The indices i and j take on the values u and d. The first term in Eq. (5) has been retained because in the lattice formulation of the theory it is essential that the fermion matrix be nonlinear in the chemical potential in order to obtain finite results in the limit of zero lattice spacing.[10] For the lattice version of the theory, we use staggered fermions and write the effective action as

$$S_{\mathrm{eff}}=S_\mathrm{W}(U)+(N_f/4)\mathrm{Tr}\ln M(U,\mu). \quad (6)$$

U is now a set of SU(3) matrices associated with the links of the lattice and $S_\mathrm{W}(U)$ is the standard Wilson action. For $N_f=2$, this action describes two flavors of equal-mass quarks at the same chemical potential. M is given by

$$M(U)_{i,j}=2ma\delta_{i,j}+\sum_{v=x,y,z}\eta_{i,v}[U_{i,v}\delta_{i,j-v}-U^\dagger_{i-v,v}\delta_{i,j+v}]+\eta_{i,t}[U_{i,t}e^{a_t\mu}\delta_{i,j-t}-U^\dagger_{i-t,t}e^{-a_t\mu}\delta_{i,j+t}]. \quad (7)$$

Here a_t is the temporal lattice spacing and η is the standard staggered fermion phase factor. The prescription for including the chemical potential in the lattice theory has been discussed by several authors.[10,11] The constraint on the prescription derived by Gavai[10] from finiteness of the energy density for finite μ is particularly important, since this condition is also necessary for the finiteness of the susceptibility.

The expressions for the susceptibilities become

$$\chi_\mathrm{S}=\frac{1}{V_s\beta}\frac{N_f}{4}\left\{\left\langle\mathrm{Tr}\frac{1}{M}\frac{\partial^2M}{\partial\mu^2}\right\rangle_U-\left\langle\mathrm{Tr}\frac{1}{M}\frac{\partial M}{\partial\mu}\frac{1}{M}\frac{\partial M}{\partial\mu}\right\rangle_U\right\}$$
$$+\frac{1}{V_s\beta}\left[\frac{N_f}{4}\right]^2\left\{\left\langle\mathrm{Tr}\frac{1}{M}\frac{\partial M}{\partial\mu}\mathrm{Tr}\frac{1}{M}\frac{\partial M}{\partial\mu}\right\rangle_U-\left\langle\mathrm{Tr}\frac{1}{M}\frac{\partial M}{\partial\mu}\right\rangle_U^2\right\}, \quad (8)$$

$$\chi_\mathrm{NS}=\frac{1}{V_s\beta}\frac{N_f}{4}\left\{\left\langle\mathrm{Tr}\frac{1}{M}\frac{\partial^2M}{\partial\mu^2}\right\rangle_U-\left\langle\mathrm{Tr}\frac{1}{M}\frac{\partial M}{\partial\mu}\frac{1}{M}\frac{\partial M}{\partial\mu}\right\rangle_U\right\}. \quad (9)$$

The fourth term in Eq. (8) is zero at $\mu=0$. The first terms in Eqs. (8) and (9) appear because μ enters nonlinearly in M. The presence of these terms is essential to obtain finite results in the continuum limit as they contain divergences that cancel against corresponding ones in the second terms. This can be seen explicitly by our computing the susceptibilities for the free theory ($U=1$) at zero temperature, where the first and second terms cancel identically. The third term in Eq. (8), which involves two traces, is zero in the free field theory and at zero temperature in the interacting theory. At finite temperature in the interacting theory, it can be seen from a hopping-parameter expansion that for a given configuration of the gauge variables $\mathrm{Tr}(1/M)\partial M/\partial\mu$ is pure imaginary, receiving contributions only from loops which wind around the lattice in the Euclidean time direction. Therefore the third term makes a negative contribution to χ_S.

Details of the simulation method are given by Gottlieb *et al.*[12] It is too costly in computer time to obtain the full inverse of M which is needed to evaluate the traces in Eqs. (8) and (9) exactly. Instead we replace these traces by unbiased estimators, by introducing a set of L Gaussian-distributed random vectors, R_i, and evaluating

$$\chi_\mathrm{S}=\frac{1}{V_s\beta}\frac{N_f}{4}\left\{\left\langle R_1^*\frac{1}{M}\frac{\partial^2M}{\partial\mu^2}R_1\right\rangle_{U,R}-\left[1+\frac{N_f}{4}\frac{1}{L}\right]\left\langle R_1^*\frac{1}{M}\frac{\partial M}{\partial\mu}\frac{1}{M}\frac{\partial M}{\partial\mu}R_1\right\rangle_{U,R}\right\}$$
$$+\frac{1}{V_s\beta}\left[\frac{N_f}{4}\frac{1}{L}\right]^2\left\langle\sum_{i,j=1}^{L}R_i^*\frac{1}{M}\frac{\partial M}{\partial\mu}R_iR_j^*\frac{1}{M}\frac{\partial M}{\partial\mu}R_j\right\rangle_{U,R}, \quad (10)$$

and

$$\chi_{\rm NS}=\frac{1}{V_s\beta}\frac{N_f}{4}\left\{\left\langle R_1^*\frac{1}{M}\frac{\partial^2 M}{\partial\mu^2}R_1\right\rangle_{U,R}-\left\langle R_1^*\frac{1}{M}\frac{\partial M}{\partial\mu}\frac{1}{M}\frac{\partial M}{\partial\mu}R_1\right\rangle_{U,R}\right\}, \tag{11}$$

where we now average over both the gauge configurations, U, and the random vectors R. The average over the R_i recovers the traces in Eqs. (8) and (9). The $1/L$ correction in the second term of Eq. (10) compensates for the unwanted contraction in the diagonal elements of the sum in the third term. The reason for our using several random vectors in the third term, which contains two traces, is that the variance of this term is many times larger than the variance of the terms containing only one trace. It is advantageous to use several random vectors to estimate it. For

$$L \ll \mathrm{Tr}\left(\frac{1}{M}\frac{\partial M}{d\mu}\right)^2\left(\mathrm{Tr}\frac{1}{M}\frac{\partial M}{\partial\mu}\right)^{-2},$$

the variance of the estimate is proportional to $1/L^2$.

Our lattice size is $8^3\times 4$. Lattices as small as this have large effects from the finite lattice spacing.[13] We have therefore evaluated the quark-number susceptibility for two flavors of free quarks with various masses on an $8^3\times 4$ lattice. For example, on the $8^3\times 4$ lattice ($T=\frac{1}{4}$) we find $\chi_{\rm S}=\chi_{\rm NS}=0.291$ for free massless quarks, in contrast to the continuum value of 0.125.

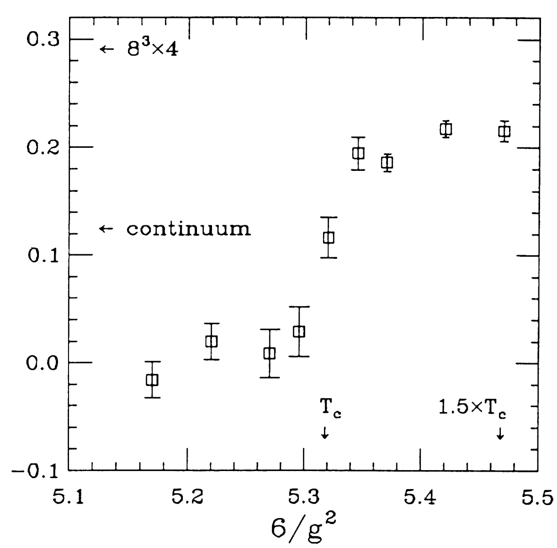

FIG. 1. $\chi_{\rm S}$ as a function of $6/g^2$ for quark mass of 0.05. Horizontal arrows label the values of $\chi_{\rm S}$ for two flavors of free quarks of this mass on an $8^3\times 4$ lattice and in the continuum. Vertical arrows indicate our earlier estimates of the values of $6/g^2$ at which the high-temperature crossover occurs for $N_T=4$ and 6. This may be interpreted as the crossover temperature and 1.5 times the crossover temperature.

Our numerical results are presented in Figs. 1–3. In Fig. 1 we show data for $\chi_{\rm S}$ for a quark mass of 0.05. In the broken-symmetry phase $\chi_{\rm S}$ is consistent with zero, as one would expect for confined quarks. The sharp rise in $\chi_{\rm S}$ as $6/g^2$ is increased through its critical value indicates that the fundamental excitations in the symmetric phase carry small free energy. In Fig. 2 we show similar data for $\chi_{\rm NS}$. The nonsinglet susceptibility need not be small in the broken-symmetry phase since it measures the ease of addition of an isospin-1 pion to the system. The fact that $\chi_{\rm NS}$ approaches approximately the same limit as $\chi_{\rm S}$ in the symmetric phase is consistent with the fundamental excitations in this phase being light mass quarks and antiquarks. These masses are significantly smaller than the masses characterizing the spatial screening of color-singlet sources.[8,9] The crossover of $\chi_{\rm S}$ and $\chi_{\rm NS}$ tracks the crossover of $\bar\psi\psi$ for this mass which is plotted in Fig. 9 of Ref. 5.

The large error bars for $\chi_{\rm S}$ in the broken-symmetry phase result from fluctuations in the two-trace term of Eq. (10). We plot this term, which forms the difference between $\chi_{\rm S}$ and $\chi_{\rm NS}$, in Fig. 3. This difference represents the off-diagonal susceptibility, or the response of the d-quark density to the u-quark chemical potential.

It will be interesting to extend our measurements to lighter quark masses. The pion mass will decrease with the quark mass. So for small values of the quark mass and $T<T_C$, one expects $\chi_{\rm NS}$ to be larger than $\chi_{\rm S}$. Work

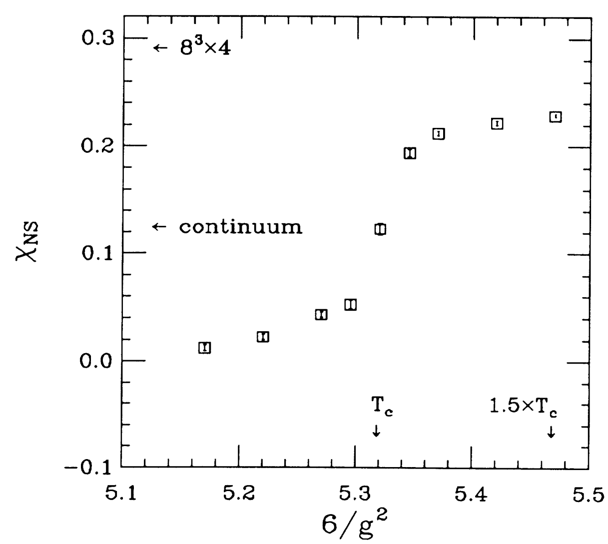

FIG. 2. $\chi_{\rm NS}$ as a function of $6/g^2$ for quark mass of 0.05.

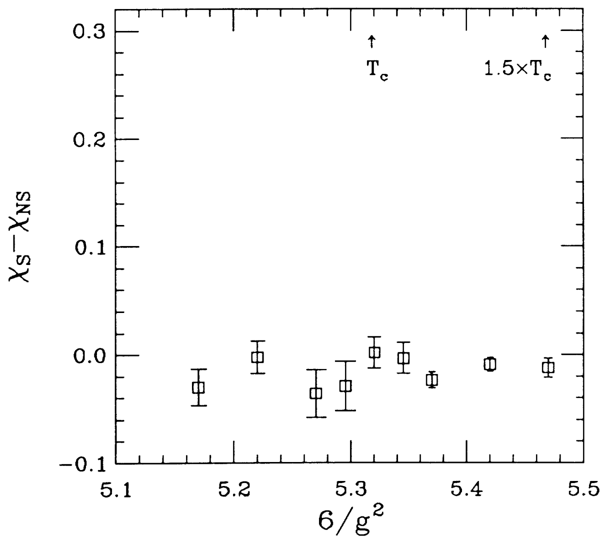

FIG. 3. $\chi_S - \chi_{NS}$ as a function of $6/g^2$ for quark mass of 0.05. This quantity is the two-trace term in Eq. (10).

in this direction is in progress.

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